

PROOF OF THE GHAHRAMANI–LAU CONJECTURE

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ABSTRACT. The Ghahramani–Lau conjecture is established; in other words, the measure algebra of every locally compact group is strongly Arens irregular. To this end, we introduce and study certain new classes of measures (called approximately invariant, respectively, strongly singular) which are of interest in their own right. Moreover, we show that the same result holds for the measure algebra of any (not necessarily locally compact) Polish group.

1. INTRODUCTION

The study of the Arens products on the second dual of a Banach algebra has been an active area of functional analysis and abstract harmonic analysis for many years. As is well-known, operator algebras (and their quotient algebras) are Arens regular, i.e., both Arens products on the second dual coincide. However, the situation is radically different for group algebras such as the convolution algebra $L_1(G)$ over a locally compact group G : building on the pioneering work (in the abelian case) of Civin–Yood [CY] from 1961, N.J. Young [Y] showed in 1973 that $L_1(G)$ is never Arens regular unless G is finite. It was thus natural to ask how irregular the multiplication in the bidual of $L_1(G)$ is – this was only settled in 1988 by Lau–Losert [LL] who showed that $L_1(G)$ is strongly Arens irregular (in the terminology established in [DL]); in other words, left multiplication by $m \in L_1(G)^{**}$ on $L_1(G)^{**}$ is the same with respect to both Arens products only if $m \in L_1(G)$, and this holds as well for right multiplication by m .

Since the measure algebra $\mathbf{M}(G)$ contains $L_1(G)$ as a closed subalgebra (in fact, as an ideal), it is clear by Young’s theorem that $\mathbf{M}(G)$ is only Arens regular for finite groups G . It was conjectured by Lau [L2] and Ghahramani–Lau [GL] that, as

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in the case of $L_1(G)$, the measure algebra $\mathbf{M}(G)$ is also strongly Arens irregular for any locally compact group G . In [N], this was established for two classes of locally compact non-compact groups: those whose cardinality is a non-measurable cardinal, and those for which the relation $\kappa(G) \geq 2^{\chi(G)}$ holds, where $\kappa(G)$ denotes the compact covering number, and $\chi(G)$ the local weight, also known as the character.

In this paper, we prove the Ghahramani-Lau conjecture for all locally compact groups.

2. MAIN RESULT AND SOME BASICS

Vector spaces are over the reals \mathbb{R} or over the complex field \mathbb{C} . The linear span of a set D in a vector space is $\text{span}(D)$. When M is a normed space, $\mathbf{B}_1(M)$ is the unit ball in M , and M^* is the dual of M . For $f \in M^*$, $\mu \in M$, the value of the functional f at μ will be written as $\langle f, \mu \rangle$. As usual, M is identified with a subspace of the second dual M^{**} by the canonical embedding (which amounts to $\langle \mu, f \rangle = \langle f, \mu \rangle$).

Now assume that M is a Banach algebra. The multiplication \star of M is extended to the *left* (or first) *Arens product* \square on M^{**} as follows, first defining a right action \cdot of M on M^* and then a left action \odot of M^{**} on M^* :

$$\begin{aligned} \langle h \cdot \mu, \nu \rangle &= \langle h, \mu \star \nu \rangle \text{ for } h \in M^*, \mu, \nu \in M, \\ \langle \mathbf{n} \odot h, \mu \rangle &= \langle \mathbf{n}, h \cdot \mu \rangle \text{ for } \mathbf{n} \in M^{**}, h \in M^*, \mu \in M, \\ \langle \mathbf{m} \square \mathbf{n}, h \rangle &= \langle \mathbf{m}, \mathbf{n} \odot h \rangle \text{ for } \mathbf{m}, \mathbf{n} \in M^{**}, h \in M^*. \end{aligned}$$

Note that if $\mu, \nu \in M$ and $h \in M^*$ then (using the embedding of M into M^{**})

$$(\#) \quad \langle \mu \square \nu, h \rangle = \langle \nu \odot h, \mu \rangle = \langle h \cdot \mu, \nu \rangle = \langle h, \mu \star \nu \rangle.$$

The (left) *topological centre* of M^{**} is defined as

$$Z_t(M^{**}) = \{ \mathbf{m} \in M^{**} : \text{the mapping } \mathbf{n} \mapsto \mathbf{m} \square \mathbf{n} \text{ is weak}^* \text{ continuous on } M^{**} \}.$$

It is easy to show that $M \subseteq Z_t(M^{**})$ (see [Da] p.248ff. for further discussion of Arens products and topological centres). There is a second canonical method to extend the multiplication of M . It gives the right (or second) Arens product. The left topological centre consists of those elements $\mathbf{m} \in M^{**}$ for which the two Arens products coincide whenever \mathbf{m} is the left factor ([Da] Def.2.6.19). In [DL] refined notions have been introduced. There, Z_t is denoted as $Z_t^{(1)}$ and one has a corresponding notion of right topological centre $Z_t^{(2)}$. Then (as the minimal case)

M is called *strongly Arens irregular* if both topological centres coincide with M ([DL] Def. 2.18).

When Ω is a locally compact topological space, $\mathbf{M}(\Omega)$ is the Banach space of bounded real or complex Radon measures on Ω with the total variation norm. $C_0(\Omega)$ denotes the space of real- or complex-valued continuous functions on Ω vanishing at infinity, equipped with supremum norm. $\mathbf{M}(\Omega)$ is identified with $C_0(\Omega)^*$ by $\langle \mu, f \rangle = \int f d\mu$ (Riesz Representation Theorem). When $\Omega = G$ is a locally compact topological group, $\mathbf{M}(G)$ is a Banach algebra with convolution \star ([Da] p. 374ff.).

Main Theorem. $Z_t(\mathbf{M}(G)^{**}) = \mathbf{M}(G)$ holds for every locally compact group G .

This was the conjecture of Ghahramani-Lau. It was proved by Lau [L1] for discrete groups, and by Neufang [N] when $\kappa(G) \geq 2^{\chi(G)}$ (and in some other cases).

Corollary. $\mathbf{M}(G)$ is always strongly Arens irregular.

Proof. The corresponding result for the right topological centre follows by using that $Z_t^{(2)}(M^{**}) = Z_t^{(1)}((M^{\text{op}})^{**})$ ([DL] p. 22), where M^{op} denotes the opposite algebra of M . Furthermore, $\mathbf{M}(G)^{\text{op}} = \mathbf{M}(G^{\text{op}})$. \square

If K is a closed subgroup (not necessarily normal) of a topological group G , then G/K denotes the space of left cosets with the quotient topology ([HR] Def. 5.15).

For set-theoretic notions, e.g. definitions about cardinal numbers, see [Je] or [K] (see also Sections 5 and 8 for comments about ordinal numbers). If Ω is a topological space, the *compact covering number* of Ω , denoted $\kappa(\Omega)$, is the least cardinal τ such that Ω is a union of τ compact subsets. $d(\Omega)$, the *density character* (shortly: density) is the least cardinal of a dense subset of Ω . For $\omega \in \Omega$ define $\chi(\omega)$ (*local weight* or character) to be the least cardinal τ such that ω has a base of neighbourhoods of cardinality τ (more accurately, one should write $\chi(\omega, \Omega)$, but if the group of homeomorphisms acts transitively on Ω it does not depend on the choice of ω ; in particular for $\Omega = G$ or $\Omega = G/K$ which are the cases used below). When K is compact, the coset space G/K is metrizable if and only if $\chi(G/K) \leq \aleph_0$ (the group case is well known: [HR, 8.3]; the general case was done by Kristensen: [HR, 8.14d]).

The cardinality of a set E is denoted by $|E|$. For any locally compact group G and a compact subgroup K of infinite index, we have $|G/K| = \kappa(G) \cdot 2^{\chi(G/K)}$ (this can be shown as in the group case, compare [HN] L. 5.4). Furthermore, we have

$d(\mathbf{M}(G/K)) = |G/K|$ (this is trivial when K is open, otherwise it follows from [HN] Thm. 5.5).

When $\mu, \nu \in \mathbf{M}(\Omega)$, $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν and $\mu \perp \nu$ means that μ and ν are mutually singular. We have $\mu \perp \nu$ if and only if $|\mu| \perp |\nu|$. When $D, D' \subseteq \mathbf{M}(\Omega)$, $D \perp D'$ means that $\mu \perp \nu$ whenever $\mu \in D$ and $\nu \in D'$. If Ω' is a locally compact subspace of Ω , any measure on Ω' has a trivial extension to Ω (defining it to be zero outside Ω'). In this way, $\mathbf{M}(\Omega')$ will be considered as a subspace of $\mathbf{M}(\Omega)$ (if Ω' is closed, this embedding is just the dual of the restriction operator $C_0(\Omega) \rightarrow C_0(\Omega')$). In particular, if H is a closed subgroup of G , we consider $\mathbf{M}(H)$ as a (weak* closed) subalgebra of $\mathbf{M}(G)$.

$\mathbf{M}_a(G)$ denotes the subspace of those measures in $\mathbf{M}(G)$ that are absolutely continuous with respect to some (left or right) Haar measure λ_G (G a locally compact group). Fixing λ_G , this will be identified in the usual way with the space $L^1(G)$ defined by λ_G . $\mathbf{M}_s(G)$ is the subspace of the measures $\mu \in \mathbf{M}(G)$ such that $\mu \perp \lambda_G$; equivalently, $\mu \perp \nu$ for every $\nu \in \mathbf{M}_a(G)$ (this notation differs from that in [HR] (19.13), where $\mathbf{M}_s(G)$ stands for the space of singular *continuous* measures). δ_x ($\in \mathbf{M}(G)$) denotes the point mass at $x \in G$.

If $\nu \in \mathbf{M}(G)$ and $h \in C_0(G)$, then (using the embedding of $C_0(G)$ into $\mathbf{M}(G)^* = C_0(G)^{**}$) we have $\nu \odot h \in C_0(G)$, defining a left action of $\mathbf{M}(G)$ on $C_0(G)$ ([Da] p.376ff., where this is denoted as $\nu \cdot h$). By (#), $\langle \nu \odot h, \mu \rangle = \langle h, \mu \star \nu \rangle$, in particular $\delta_x \odot h(y) = h(yx)$ for $x, y \in G$ (right translation of functions).

If K is a compact subgroup of G , $\lambda_K \in \mathbf{M}(K) \subseteq \mathbf{M}(G)$ shall always be its normalized Haar measure (i.e., $\lambda_K(K) = 1$). Recall that $\lambda_K \star \lambda_K = \lambda_K$ and for those readers who are more acquainted to classical convolution, we mention that (using invariance of λ_K under the group inversion) one has $\lambda_K \odot h = h \star \lambda_K$ (but, in principle, all our results about \odot can be proved using duality arguments). If K is normal in G , then λ_K is central in $\mathbf{M}(G)$ (and conversely).

Lemma 1. *For G a locally compact group and a compact subgroup K of G put $\mathbf{M}(G, K) = \{\mu \in \mathbf{M}(G) : \mu = \mu \star \lambda_K\} = \mathbf{M}(G) \star \lambda_K$. Then the canonical mapping $\pi: G \rightarrow G/K$ induces an isometric isomorphism between $\mathbf{M}(G, K)$ and $\mathbf{M}(G/K)$.*

Before giving the proof, we state some conventions that will be important throughout the paper. Using this isomorphism as identification, $\mathbf{M}(G/K)$ will be considered as a weak* closed subspace (left ideal) of $\mathbf{M}(G)$ (this corresponds to the attitude of [Di, 22.6]). Note that if $x \in G$ normalizes K (i.e., $xKx^{-1} = K$), we have

$\mathbf{M}(G/K) \star \delta_x = \mathbf{M}(G/K)$. If K is normal, the multiplication on $\mathbf{M}(G/K)$ inherited from $\mathbf{M}(G)$ corresponds to convolution defined by the quotient group G/K . Similarly, $C_0(G/K)$ will be identified with $\lambda_K \odot C_0(G)$ (right K -periodic functions in $C_0(G)$). One can use this also to define the subspaces $\mathbf{M}_a(G, K) = \mathbf{M}_a(G) \cap \mathbf{M}(G/K)$ and $\mathbf{M}_s(G, K) = \mathbf{M}_s(G) \cap \mathbf{M}(G/K)$. Alternatively, there exists a left G -invariant measure $\lambda_{G/K}$ on G/K (e.g. by [HR, Thm. 15.24]) and $\mathbf{M}_a(G, K)$ (resp. $\mathbf{M}_s(G, K)$) consist of the measures $\mu \in \mathbf{M}(G/K)$ with $\mu \ll \lambda_{G/K}$ (resp. $\mu \perp \lambda_{G/K}$).

Proof. This is contained in the proof of Thm. 8.1B of [Jw] (in a different notation). We include a direct argument. $f \mapsto f \circ \pi$ defines an isometric linear mapping $C_0(G/K) \rightarrow C_0(G)$. Its image consists of all right K -periodic functions in $C_0(G)$ (i.e., $h = f \circ \pi$ satisfies $h(yx) = h(y)$ for all $x \in K, y \in G$). For general $h \in C_0(G)$ we have $\lambda_K \odot h(y) = \langle \lambda_K \odot h, \delta_y \rangle = \int h(yx) d\lambda_K(x)$. It follows easily that right periodicity of h is equivalent to $h = \lambda_K \odot h$. Since λ_K is idempotent, we can see that the image of the embedding coincides with $\lambda_K \odot C_0(G)$.

For $\mu \in \mathbf{M}(G)$ the *image measure* $\dot{\mu} \in \mathbf{M}(G/K)$ is defined by $\dot{\mu}(B) = \mu(\pi^{-1}(B))$. Easy computations show that $\mu \mapsto \dot{\mu}$ is just the dual mapping of the embedding of C_0 described above and that $(\mu \star \lambda_K)^\cdot = \dot{\mu}$. Finally, one can conclude (from the properties at the C_0 -level) that the restriction to $\mathbf{M}(G, K)$ is isometric and surjective. Alternatively: the inverse mapping from $\mathbf{M}(G/K)$ to $\mathbf{M}(G, K)$ is a special case of the mapping $m \mapsto m^\sharp$ investigated in [B, Chap. 7, § 2] (the dual mapping of [HR, Thm. 15.21]). \square

The following notion will provide an essential tool for our argument.

Definition. Let τ be a cardinal. Say that a measure $\mu \in \mathbf{M}(G)$ is τ -thin if there is a set $P \subseteq G$ such that $|P| = \tau$ and $\mu \star \delta_p \perp \mu \star \delta_{p'}$ for all $p, p' \in P$ with $p \neq p'$.

The basic observation used in [N] was that $Z_t(M^{**})$ must be small whenever there exists $h \in M^*$ such that $\mathbf{B}_1(M^{**}) \odot h$ is big. A general version is worked out in Section 3, see the Lemmas 3 and 5 for precise conditions. This motivates the search for factorization theorems. In the case of $\mathbf{M}(G)$ it has turned out to be very difficult to get information about the behaviour of $\mathbf{n} \odot h$ for general $\mathbf{n} \in M^{**}$. Thus we have restricted to $\mathbf{n} = \delta_x$ ($x \in G$) and limits $\mathbf{n} \in \overline{\delta(G)}$ (we write $\delta(G) = \{\delta_x : x \in G\}$, $\overline{}$ denotes the weak* closure in the bidual). For this case, factorizations are constructed in Section 4, using thinness of measures (Theorems 10 and 12).

It is almost immediate that if $\kappa(G)$ is uncountable, any measure $\mu \in \mathbf{M}(G)$ is $\kappa(G)$ -thin (recall that the support of μ is always contained in a σ -compact subgroup) and this was used in [N] to prove the conjecture in the case where $\kappa(G) \geq 2^{\chi(G)}$. But this approach is insufficient when $\kappa(G)$ is small (in particular for compact G). In Section 5, we consider the case of singular measures, where stronger conclusions are possible. Extending a technique of [P] to general locally compact groups, we will show (Theorem 15) that if G is non-discrete, any $\mu \in \mathbf{M}_s(G)$ is $2^{\aleph_0} \kappa(G)$ -thin. Using this, we can prove in Section 6 the Main Theorem for metrizable groups (and also for all G with $|G| \leq 2^{\aleph_0} \kappa(G)$), see Theorem 17. To make this important case more easily accessible, we provide some “light” versions of the preliminary results (Corollaries 4 and 11).

For the proof of the Main Theorem in the general case, the basic idea is to consider subspaces $\mathbf{M}(G/K)$ (K a compact subgroup) and to use induction on $\chi(G/K)$. In Section 7 we collect first some results on the behaviour of $\chi(G/K)$ and the possibilities to replace K by a normal subgroup. In the non-metrizable case a further refinement of the decomposition of $\mathbf{M}(G)$ is introduced (Theorem 22). Instead of singular measures we consider “strongly singular measures”. In Section 8 we show that any $\mu \in \mathbf{M}_{ss}(G, K)$ is $|G/K|$ -thin (Corollary 31). Then in Section 9 the final work is done, proving Theorem 40 which contains our Main Theorem.

3. SUBSPACES AND DIRECT SUMS

Let M be a Banach space, M_1 a closed subspace. $M_1^\circ \subseteq M^*$ shall denote the annihilator of M_1 (the continuous functionals vanishing on M_1). Recall that M_1° can be identified with the dual space $(M/M_1)^*$, the weak* topology being just the induced topology from M^* . The bidual M_1^{**} can be identified with a subspace of M^{**} (using the second dual of the inclusion mapping) and this is compatible with the embedding of M_1 into its bidual. Then, M_1^{**} is just the weak* closure of M_1 in M^{**} and $M_1^{**} = M_1^{\circ\circ}$ (where the second annihilator refers to M^{**}).

Lemma 2. *Let M be a Banach algebra, M_1 a closed subspace.*

- (1) *If $\mu \in \mathbf{B}_1(M)$ is such that $M_1 \star \mu \subseteq M_1$, then $\mu \odot \mathbf{B}_1(M_1^\circ) \subseteq \mathbf{B}_1(M_1^\circ)$ and $\mathbf{B}_1(M_1^{**}) \square \mu \subseteq \mathbf{B}_1(M_1^{**})$.*
- (2) *For every $h \in M^*$ the mapping $\mathfrak{m} \mapsto \mathfrak{m} \odot h$ from M^{**} to M^* is continuous for the weak* topologies on M^{**} and M^* .*

Proof. This follows from the definitions of \odot and \square . \square

Lemma 3. *Let M be a Banach algebra, M_1 a closed subspace, and assume that there exists $h \in M^*$ such that $\mathbf{B}_1(M^{**}) \odot h \supseteq \mathbf{B}_1(M_1^\circ)$. Then $Z_t(M^{**}) \subseteq M + M_1^{**}$.*

Proof. Take $\mathbf{m} \in Z_t(M^{**})$. The idea is to show that the assumptions of the lemma imply weak* continuity of the restriction of \mathbf{m} to M_1° . Then (recall that the isomorphism of M_1° and $(M/M_1)^*$ respects the weak* topologies) there exists $\mu \in M$ such that $\langle \mathbf{m}, u \rangle = \langle u, \mu \rangle$ holds for all $u \in M_1^\circ$. Consequently, $\mathbf{m} - \mu \in M_1^{\circ\circ} = M_1^{**}$.

To show weak* continuity, we consider the functional $\psi: \mathbf{n} \mapsto \langle \mathbf{m}, \mathbf{n} \odot h \rangle$ on M^{**} . ψ is weak* continuous, since $\langle \mathbf{m}, \mathbf{n} \odot h \rangle = \langle \mathbf{m} \square \mathbf{n}, h \rangle$ and $\mathbf{m} \in Z_t(M^{**})$. Let τ_1 be the quotient topology on $\mathbf{B}_1(M^{**}) \odot h$ obtained from the weak* topology on $\mathbf{B}_1(M^{**})$ by the mapping $\mathbf{n} \mapsto \mathbf{n} \odot h$. Weak* continuity of ψ implies that \mathbf{m} is τ_1 -continuous. Using compactness and part (2) of Lemma 2 one can see that τ_1 coincides with the topology on $\mathbf{B}_1(M^{**}) \odot h$ induced by the weak* topology of M^* . By our assumption on h , this implies that $\mathbf{m}|_{\mathbf{B}_1(M_1^\circ)}$ is weak* continuous. Then, by the Krein-Šmulian (or Banach-Dieudonné) theorem, $\mathbf{m}|_{M_1^\circ}$ is weak* continuous. \square

Corollary 4. *Assume that $M = M_0 \oplus M_1$ is the topological direct sum of closed subspaces and M_1 satisfies the assumption of Lemma 3. Then $Z_t(M^{**}) \subseteq M_0 \oplus M_1^{**}$.*

We have $M/M_1 \cong M_0$ (hence $M_1^\circ \cong M_0^*$). Here \cong shall mean that these are isomorphic Banach spaces, but not necessarily isometric.

For the proof of our main theorem in the non-metrizable case, we need another variation of Lemma 3. If M_2 is a closed subspace of M , $p_2: M^* \rightarrow M^*/M_2^\circ$ denotes the canonical projection. When M^*/M_2° is identified with M_2^* , this is the dual map of the inclusion $M_2 \rightarrow M$ (hence weak* continuous). It assigns to a functional $h \in M^*$ its restriction $h|_{M_2}$ to M_2 .

Lemma 5. *Let M be a Banach algebra, $M_1 \subseteq M_2$ closed subspaces of M and assume that there exists $h \in M^*$ such that $p_2(\mathbf{B}_1(M^{**}) \odot h) \supseteq p_2(\mathbf{B}_1(M_1^\circ))$. Then $Z_t(M^{**}) \cap M_2^{**} \subseteq M_2 + M_1^{**}$.*

Proof. This is similar to the proof of Lemma 3. Take $\mathbf{m} \in Z_t(M^{**}) \cap M_2^{**}$. Since $\mathbf{m} \in M_2^{**}$, it induces a linear functional \mathbf{m}' on M^*/M_2° , satisfying $\mathbf{m} = \mathbf{m}' \circ p_2$. Considering ψ as in Lemma 3 and the quotient topology on $p_2(\mathbf{B}_1(M^{**}) \odot h)$ arising from the composed mapping $\mathbf{n} \mapsto \mathbf{n} \odot h \mapsto p_2(\mathbf{n} \odot h)$, the assumption $\mathbf{m} \in Z_t(M^{**})$ and the condition on h imply as above weak* continuity of $\mathbf{m}'|_{p_2(M_1^\circ)}$.

Then there exists $\mu \in M_2$ such that $\langle \mathbf{m}', u \rangle = \langle u, \mu \rangle$ holds for all $u \in p_2(M_1^\circ)$ and our claim follows. \square

4. FACTORIZATION FOR THIN MEASURES

The purpose of this section is to prove Theorem 10 and Corollary 11, which are key steps in the proof of the main result. An extended version will be given in Theorem 12.

For $\mu \in \mathbf{M}(G)$ and $x \in G$ write $\mu \star x = \mu \star \delta_x$. Thus $\mu \star x(Bx) = \mu(B)$ when $B \subseteq G$ is a Borel set (or $|\mu|$ -measurable). When $H \subseteq G$, write $\mu \star H = \{\mu \star h : h \in H\}$. When $D \subseteq \mathbf{M}(G)$ and $H \subseteq G$, write $D \star H = \{\mu \star h : \mu \in D, h \in H\}$. Recall that $\delta(G) = \{\delta_x : x \in G\}$, $\overline{\delta(G)}$ denotes the weak* closure in $\mathbf{M}(G)^{**}$, giving a weak* compact subset of its unit ball.

Lemma 6. *Let G be any locally compact group. Let $\{D_\gamma : \gamma \in \Gamma\}$ be a family of subspaces of $\mathbf{M}(G)$ such that $D_\beta \perp D_\gamma$ when $\beta, \gamma \in \Gamma$, $\beta \neq \gamma$. If $h_\gamma \in \mathbf{B}_1(D_\gamma^*)$ is given for each $\gamma \in \Gamma$, then there exists $h \in \mathbf{B}_1(\mathbf{M}(G)^*)$ that agrees with h_γ on D_γ for every $\gamma \in \Gamma$.*

As a special case (taking $D_x = \text{span}\{\delta_x\}$ for $x \in G$), one can see that the set $\delta(G)$ is weakly discrete in $\mathbf{M}(G)$ and thus weak* discrete in $\mathbf{M}(G)^{**}$.

Proof. If μ is in the space D' spanned by $\bigcup_\gamma D_\gamma$, then $\mu = \sum_{k=1}^n \mu_k$ where $\mu_k \in D_{\gamma_k}$ and $\gamma_j \neq \gamma_k$ for $j \neq k$. There is then a unique linear functional h' on D' extending each h_D . Note that if $\mu', \mu'' \in \mathbf{M}(G)$ and $\mu' \perp \mu''$ then $\|\mu' + \mu''\| = \|\mu'\| + \|\mu''\|$ and hence

$$|\langle h', \mu \rangle| \leq \sum_k |\langle h', \mu_k \rangle| \leq \sum_k \|h_{\gamma_k}\| \|\mu_k\| \leq \sum_k \|\mu_k\| = \|\mu\|.$$

This shows that the norm of h' is at most 1 and thus h' extends to a linear functional $h \in \mathbf{B}_1(\mathbf{M}(G)^*)$. \square

Lemma 7. *Let G be any locally compact group, τ uncountable. Assume that M_0 is a subspace of $\mathbf{M}(G)$ such that if $\mu \in M_0$ then $|\mu| \in M_0$ and μ is τ -thin. If $F \subseteq M_0$ is finite, $D \subseteq \mathbf{M}(G)$ and $|D| < \tau$, then there exists $x \in G$ such that $D \perp (F \star x)$.*

Proof. The measure $\nu = \sum_{\xi \in F} |\xi|$ is τ -thin, hence there is a set $P \subseteq G$ such that $|P| = \tau$ and $\nu \star p \perp \nu \star p'$ for all $p, p' \in P$ with $p \neq p'$. For every $\mu \in D$ the set $P_\mu = \{p \in P : \mu \text{ is not singular to } \nu \star p\}$ must be countable, because μ is a finite

measure. Thus $|\bigcup_{\mu \in D} P_\mu| \leq |D| \cdot \aleph_0 < \tau = |P|$, so that there exists $x \in P \setminus \bigcup_{\mu \in D} P_\mu$. Then $D \perp \nu \star x$ and therefore $D \perp (F \star x)$. \square

Lemma 8. *Let G be any locally compact group, τ uncountable. Assume that M_0 is a subspace of $\mathbf{M}(G)$ such that if $\mu \in M_0$ then $|\mu| \in M_0$ and μ is τ -thin. If $\{F_\gamma : \gamma \in \Gamma\}$ is a family of finite subsets of M_0 and $|\Gamma| \leq \tau$, then there exist $x_\gamma \in G$ for $\gamma \in \Gamma$ such that $(F_\beta \star x_\beta) \perp (F_\gamma \star x_\gamma)$ when $\beta, \gamma \in \Gamma$, $\beta \neq \gamma$.*

Proof. Construct x_γ by transfinite induction, using Lemma 7 at each step. \square

We say that a direct sum $M_2 = M_0 \oplus M_1$ of subspaces of $\mathbf{M}(G)$ is G -invariant, if $M_i \star G \subseteq M_i$ holds for $i = 0, 1$ (which implies $M_2 \star G \subseteq M_2$).

Lemma 9. *Let G be a locally compact group, $M_2 = M_0 \oplus M_1$ a G -invariant topological direct sum of closed subspaces of $\mathbf{M}(G)$. Let \mathcal{O} be a collection of weak* open subsets of M_0^* , each of them having non-empty intersection with $\mathbf{B}_1(M_0^*)$. $\tau = |\mathcal{O}|$ shall be uncountable. Assume that $\mu \in M_0$ implies that $|\mu| \in M_0$ and that μ is τ -thin. Then there exists $h \in M_1^\circ$ such that the (projected) orbit $p_0(\delta(G) \odot h)$ intersects every set from \mathcal{O} and $h|M_0 \in \mathbf{B}_1(M_0^*)$.*

As above, $p_0 : \mathbf{M}(G)^* \rightarrow \mathbf{M}(G)^*/M_0^\circ$ denotes the canonical projection. Under the identification of $\mathbf{M}(G)^*/M_0^\circ$ with M_0^* , we have $p_0(h) = h|M_0$ (restriction of the functional).

Proof. Without loss of generality assume that each $U \in \mathcal{O}$ is a basic neighbourhood of the form

$$U = \{f \in M_0^* : |\langle f - g_U, \mu \rangle| < \varepsilon_U \text{ for all } \mu \in F_U\}$$

where $F_U \subseteq M_0$ is a finite set, $g_U \in \mathbf{B}_1(M_0^*)$ and $\varepsilon_U > 0$. Apply Lemma 8 with $\Gamma = \mathcal{O}$ to obtain elements $x_U \in G$ for $U \in \mathcal{O}$, such that if $U, V \in \mathcal{O}$, $U \neq V$, then $(F_U \star x_U) \perp (F_V \star x_V)$.

Then apply Lemma 6, taking $\Gamma = \mathcal{O}$, D_U the space spanned by $F_U \star x_U$ and the functionals $h_U \in \mathbf{B}_1(D_U^*)$ defined by $\langle h_U, \nu \rangle = \langle g_U, \nu \star x_U^{-1} \rangle$. Thus there is $h \in \mathbf{B}_1(M_0^*)$ that agrees with h_U on D_U for every $U \in \mathcal{O}$ and we may extend it to M_2 so that $h = 0$ on M_1 . Extending h further to $\mathbf{M}(G)$, this means that $h \in M_1^\circ$, $h|M_0 \in \mathbf{B}_1(M_0^*)$ and for $\mu \in F_U$, we have

$$\langle \delta_{x_U} \odot h, \mu \rangle = \langle h, \mu \star x_U \rangle = \langle h_U, \mu \star x_U \rangle = \langle g_U, \mu \star x_U \star x_U^{-1} \rangle = \langle g_U, \mu \rangle,$$

hence $p_0(\delta_{x_U} \odot h) \in U$. \square

Remark. The argument gets somewhat more transparent if $M_0 \perp M_1$ (which will always be the case in the applications below). Then M_0^* is *isometrically* isomorphic to $p_2(M_1^\circ)$ (these are the functionals on M_2 vanishing on M_1) and the construction above gives $h \in \mathbf{B}_1(M_1^\circ)$.

The method of proof above shows a slightly stronger statement: Assume that given are finite dimensional subspaces D_γ of M_0 and functionals $h_\gamma \in \mathbf{B}_1(D_\gamma^*)$ for $\gamma \in \Gamma$ such that $\tau = |\Gamma|$ is uncountable, and M_0, M_1 are as in the lemma. Then there exists $h \in M_1^\circ$ with $h|_{M_0} \in \mathbf{B}_1(M_0^*)$ and such that for every $\gamma \in \Gamma$ there exists $h'_\gamma \in \delta(G) \odot h$ satisfying $h'_\gamma|_{D_\gamma} = h_\gamma$.

Theorem 10 (Factorization for thin measures). *Let G be a locally compact group, $\mathbf{M}(G) = M_0 \oplus M_1$ a G -invariant topological direct sum such that $d(M_0)$ is uncountable. Assume that $\mu \in M_0$ implies that $|\mu| \in M_0$ and that μ is $d(M_0)$ -thin. Then there exists $h \in M_1^\circ$ such that $\overline{\delta(G)} \odot h \supseteq \mathbf{B}_1(M_1^\circ)$.*

In fact, our argument produces an $h \in \mathbf{B}_1(M_0^*)$ with $\overline{\delta(G)} \odot h = \mathbf{B}_1(M_0^*)$.

Proof. $d(M_0)$ refers to the norm topology. Taking a norm-dense subset of M_0 with cardinality $d(M_0)$, one can find a family \mathcal{O} of weak* open subsets of M_0^* whose intersections with $\mathbf{B}_1(M_0^*)$ give a weak* open base and such that $|\mathcal{O}| = d(M_0)$. Now take $h \in M_1^\circ$ as given by Lemma 9. The set $\overline{\delta(G)} \odot h$ is weak* closed by Lemma 2. Note that $h|_{M_0} \in \mathbf{B}_1(M_0^*)$ implies $p_0(\delta(G) \odot h) \subseteq \mathbf{B}_1(M_0^*)$ as in part (1) of Lemma 2. Then it follows easily from the properties of h and \mathcal{O} that $p_0(\overline{\delta(G)} \odot h) = \mathbf{B}_1(M_0^*)$.

For $h_1 \in \mathbf{B}_1(M_1^\circ)$, we have $p_0(h_1) \in \mathbf{B}_1(M_0^*)$. Hence there exists $h_2 \in \overline{\delta(G)} \odot h$ such that $p_0(h_2) = p_0(h_1)$. G -invariance of M_1 implies $h_2 \in M_1^\circ$ and it follows that $h_2 = h_1$. \square

Corollary 11. *Let G be a locally compact group, $\mathbf{M}(G) = M_0 \oplus M_1$ a G -invariant topological direct sum of closed subspaces such that $d(M_0)$ is uncountable. Assume that $\mu \in M_0$ implies that $|\mu| \in M_0$ and that μ is $d(M_0)$ -thin. Then one gets $Z_t(\mathbf{M}(G)^{**}) \subseteq M_0 \oplus M_1^{**}$.*

Proof. This follows immediately from Corollary 4 in combination with Theorem 10. \square

Remarks. In this section, we use several times the assumption that $\mu \in M_0$ implies $|\mu| \in M_0$. For a closed subspace M_0 of $\mathbf{M}(G)$ this is in fact equivalent to M_0 being a vector sublattice (i.e., in the complex case, $\mu \in M_0$ implies that its real and imaginary

part belong to M_0 and the set of real measures in M_0 is closed under the lattice operations).

In our paper, thinness of measures is a strong tool for non-separable spaces. Formally, Theorem 10 and Corollary 11 stay true without the condition that $d(M_0)$ should be uncountable (i.e., M_0 non-separable). But observe that there are no non-zero *separable* G -invariant closed sublattices M_0 in $\mathbf{M}(G)$ such that every μ in M_0 is $d(M_0)$ -thin! (Let $\{\mu_n : n \geq 0\}$ be a dense subset of $\mathbf{B}_1(M_0)$ and put $\mu = \sum_{n=0}^{\infty} |\mu_n|/2^{n+1}$. Then it is easy to see that $\nu \ll \mu$ holds for every $\nu \in M_0$. Thus, by G -invariance of M_0 , it follows that μ is not even 2-thin, unless $\mu = 0$). See also the comment following Theorem 15 below.

Again there is an extended version, needed for the proof of our Main Theorem in the non-metrizable case.

Theorem 12 (Factorization on subspaces). *Let G be a locally compact group, $M_2 = M_0 \oplus M_1$ a topological direct sum of closed subspaces of $\mathbf{M}(G)$ and let $\tau \geq d(M_0)$ be uncountable. Assume that $\mu \in M_0$ implies that $|\mu| \in M_0$ and that μ is τ -thin. Furthermore, assume that there exists a G -invariant topological direct sum $\widetilde{M}_2 = \widetilde{M}_0 \oplus \widetilde{M}_1$ of closed subspaces of $\mathbf{M}(G)$ such that $M_i \subseteq \widetilde{M}_i$ holds for $i = 0, 1$. Then there exists $h \in M_1^\circ$ such that $p_2(\overline{\delta(G)} \odot h) \supseteq p_2(\mathbf{B}_1(M_1^\circ))$ and it follows that $Z_t(\mathbf{M}(G)^{**}) \cap M_2^{**} \subseteq M_0 \oplus M_1^{**}$.*

Proof. This is similar as above. There is a canonical projection $q : (\widetilde{M}_0)^* \rightarrow M_0^*$, using restriction. As above, take a family \mathcal{O} of weak* open subsets of M_0^* whose intersections with $\mathbf{B}_1(M_0^*)$ give a weak* open base and such that $|\mathcal{O}| = \tau$. Put $\widetilde{\mathcal{O}} = \{q^{-1}(U) : U \in \mathcal{O}\}$. Now take $h \in (\widetilde{M}_1)^\circ \subseteq M_1^\circ$ obtained from Lemma 9, applied to $\widetilde{\mathcal{O}}$ and $\widetilde{M}_2 = \widetilde{M}_0 \oplus \widetilde{M}_1$. It follows immediately that the projected orbit $p_0(\delta(G) \odot h)$ intersects every set from \mathcal{O} and as above this implies $p_2(\overline{\delta(G)} \odot h) \supseteq p_2(\mathbf{B}_1(M_1^\circ))$. The conclusion on Z_t follows now from Lemma 5. \square

5. SEPARATION OF SINGULAR MEASURES

Let G be a locally compact group. Then $\mathbf{M}_s(G) \star G \subseteq \mathbf{M}_s(G)$ and $\mathbf{M}_a(G) \star G \subseteq \mathbf{M}_a(G)$. In view of the Lebesgue decomposition [HR, 19.20], we have $\mathbf{M}(G) = \mathbf{M}_s(G) \oplus \mathbf{M}_a(G)$. Lemma 9 and the other results of Section 4 apply whenever every measure in $\mathbf{M}_s(G)$ is $|G|$ -thin (recall from the introduction that $|G| = d(\mathbf{M}(G))$ holds for all infinite groups G); in this section we show that this is the case when

G is metrizable or more generally, when $|G| = 2^{\aleph_0} \kappa(G)$. Further applications will follow for the subspaces $\mathbf{M}_s(G, K)$ of $\mathbf{M}(G)$.

Versions of the following lemma are well-known. Saks ([S], III.11) gives a proof for $G = \mathbb{R}^n$ and provides references to original sources. A stronger version for $G = \mathbb{T}$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group) is proved by Prokaj [P, Th. 1]. If G is a discrete group then $\mathbf{M}_s(G)$ is the null space $\{0\}$. Thus the lemma is of interest only for non-discrete groups, although formally it holds for discrete groups as well.

Lemma 13. *Let G be a locally compact group. If $\mu \in \mathbf{M}_s(G)$ and U is any compact neighbourhood of e_G then $\mu \perp (\mu \star x)$ for λ_G -almost all x in U .*

Proof. Since the support of μ is always σ -compact, we may (replacing G by some open subgroup) assume σ -compactness of G . Put $\lambda = \lambda_G$. Since $\mu \perp \lambda$ if and only if $|\mu| \perp \lambda$, we may assume that $\mu \geq 0$ and $\mu \neq 0$.

There is a \mathbf{G}_δ -set $E \subseteq G$ such that $\mu(G \setminus E) = 0$ and $\lambda(E) = 0$. Define

$$f: G \times G \rightarrow [0, 1] \quad \text{by} \quad f(x, y) = \begin{cases} 0 & \text{if } yx \notin E \\ 1 & \text{if } yx \in E \end{cases}$$

As E is a \mathbf{G}_δ -set, the function f is Borel measurable on $G \times G$.

Then $0 \leq \int_U f(x, y) d\lambda(x) = \lambda(U \cap y^{-1}E) \leq \lambda(y^{-1}E) = 0$ for every $y \in G$, because $\lambda(E) = 0$. On the other hand, we have $\int_G f(x, y) d\mu(y) = \mu(Ex^{-1}) = \mu \star x(E)$ for every $x \in G$. Now apply Fubini's theorem for non-negative functions (being valid in the σ -compact case [HR, Thm. 13.9]) to get

$$\int_U \mu \star x(E) d\lambda(x) = \int_U \int_G f(x, y) d\mu(y) d\lambda(x) = \int_G \int_U f(x, y) d\lambda(x) d\mu(y) = 0.$$

This implies that $\mu \star x(E) = 0$ for λ -almost all x in U . Clearly, $\mu \perp (\mu \star x)$ for every x for which $\mu \star x(E) = 0$. \square

Lemma 14. *Let G be any locally compact group. Let $\mu \in \mathbf{M}(G)$, $\varepsilon > 0$, and let $H \subseteq G$ be a countable set such that the measures $\mu \star h$ are pairwise mutually singular for $h \in H$. Then there is a compact set $C \subseteq G$ such that $|\mu|(G \setminus C) < \varepsilon$ and the sets Ch are pairwise disjoint for $h \in H$.*

Proof. Using singularity, we get pairwise disjoint sets $E_h \subseteq G$ ($h \in H$) such that $\mu \star h$ is concentrated on E_h for all $h \in H$ and $|\mu \star h'| (E_h) = 0$ for all $h, h' \in H$ with $h \neq h'$.

Put $E = \bigcap_{h \in H} E_h h^{-1}$. Since $|\mu|(G \setminus E_h h^{-1}) = |\mu \star h|(G \setminus E_h) = 0$, it follows that $|\mu|(G \setminus E) = 0$. Thus there is a compact set $C \subseteq E$ such that $|\mu|(G \setminus C) < \varepsilon$. The sets Ch are pairwise disjoint for $h \in H$ because $Ch \subseteq E_h$. \square

Remark. All that is actually needed in Lemma 14 is that H has cardinality less than the additivity of the measure μ — the least cardinal of a family of null sets whose union is not a null set.

In the next proof (and further on in Theorem 30), the formalism of ordinals will be used in the way developed by von Neumann — in particular, every ordinal is equal to the set of its predecessors. For example, this means that for ordinals α and β the assertions $\alpha < \beta$, $\alpha \subsetneq \beta$ and $\alpha \in \beta$ have the same meaning. In particular, for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, the equality $n = \{0, 1, 2, \dots, n-1\}$ holds.

The following result was proved by Prokaj [P, Theorem 10] for $G = \mathbb{R}$.

Theorem 15. *Let G be any non-discrete locally compact group, $\mu \in \mathbf{M}_s(G)$. Then there exists a \mathbf{K}_σ -set $E \subseteq G$ and a set $P \subseteq G$ such that μ is concentrated on E , $|P| = 2^{\aleph_0} \kappa(G)$ and $(Ep) \cap (Ep') = \emptyset$ for all $p, p' \in P$ with $p \neq p'$. Thus, every $\mu \in \mathbf{M}_s(G)$ is $2^{\aleph_0} \kappa(G)$ -thin.*

In particular, we recover the result of [La]: if $\mu \in \mathbf{M}(G)$ and the orbit $\{\mu \star x : x \in G\}$ is (norm-) separable, then $\mu \in \mathbf{M}_a(G)$. See also [Gli] for related results. In fact, this shows once again that in the previous section our permanent assumption “ $d(M_0)$ uncountable” makes no restriction (even when M_0 is not a sublattice), as long as singular measures are considered.

If G is metrizable, then (as in [P]) the set P can be chosen to be perfect (in the proof below, just add the requirement that the diameter of the sets U_j be less than $1/j$ for $j > 0$).

Proof. For $x_j \in G$ and a finite subset $d = \{j_0, \dots, j_l\}$ of \mathbb{N} , where $j_0 < j_1 < \dots$, we define $d_* = x_{j_0} \cdots x_{j_l}$ (with $d_* = e_G$ when d is empty).

Let $\mu \in \mathbf{M}_s(G)$ and assume, without loss of generality, that $\mu \geq 0$ and $\mu \neq 0$. For $j \in \mathbb{N}$, we will construct by induction

- compact neighbourhoods U_j of e_G
- compact sets $C_j \subseteq G$
- elements $x_j \in G$,

so that the following conditions (1°) – (5°) hold:

- (1°) $(\mu \star d_*) \perp (\mu \star d'_*)$ whenever d and d' are distinct subsets of j ,
- (2°) $\mu(G \setminus C_j) \leq 2^{-j}$,
- (3°) $(C_j d_* U_{j+1}) \cap (C_j d'_* U_{j+1}) = \emptyset$ whenever d and d' are distinct subsets of j ,
- (4°) $U_{j+1} \cap (x_j U_{j+1}) = \emptyset$,
- (5°) $U_{j+1} \cup (x_j U_{j+1}) \subseteq U_j$.

Let U_0 be any compact neighbourhood of e_G . Let $n \in \mathbb{N}$ be such that U_n has been constructed as well as C_j and x_j for all $j < n$ so that (1°) – (5°) hold.

If $n = 0$, put $\nu = \mu$, otherwise $\nu = \sum_{d \subseteq n} \mu \star d_*$. Note that $\nu \in \mathbf{M}_s(G)$ and, by Lemma 13, there is x_n in the interior of U_n such that $\nu \perp (\nu \star x_n)$. From that and from (1°) for $j < n$ it follows that (1°) holds for $j = n + 1$ as well.

By Lemma 14 there is a compact set $C_n \subseteq G$ such that (2°) holds for $j = n$ and

$$(C_n d_*) \cap (C_n d'_*) = \emptyset \quad \text{for distinct } d, d' \subseteq n + 1.$$

Thus $\{C_n d_* : d \subseteq n + 1\}$ is a finite family of pairwise disjoint compact sets, and there is a neighbourhood W of e_G such that

$$(C_n d_* W) \cap (C_n d'_* W) = \emptyset \quad \text{for distinct } d, d' \subseteq n + 1.$$

Since x_n is in the interior of U_n and $x_n \neq e_G$, there is a compact neighbourhood U_{n+1} of e_G such that $U_{n+1} \subseteq W$ and (4°) and (5°) hold with $j = n$. Since $U_{n+1} \subseteq W$, we get also (3°) for $j = n$. The construction of U_j , C_j and x_j satisfying (1°) – (5°) is complete.

Next, for each $n \in \mathbb{N}$ define $E_n = \bigcap_{j=n}^{\infty} C_j$ and put $E = \bigcup_n E_n$. Then E is a \mathbf{K}_σ -set and $\mu(G \setminus E) = 0$. For $d \subseteq \mathbb{N}$ (not necessarily finite), we define $K(d) = \bigcap_{n=1}^{\infty} (d \cap n)_* U_n$.

Note (since (5°) implies $(d \cap n)_* U_n \subseteq (d \cap m)_* U_m$ for $m < n$) that $K(d) \subseteq U_0$ is nonempty, being the monotone intersection of compact sets. From (4°) it follows $K(d) \cap K(d') = \emptyset$ for $d \neq d'$. Form P_0 by taking one element in each $K(d)$. Thus $|P_0| = 2^{\aleph_0}$. Let H be the subgroup generated by $E U_0$. We take P_1 a set of representatives for the right cosets in G (with respect to H) and put $P = P_0 P_1$. Since H is open and σ -compact, we have $|P_1| = \kappa(G)$ when G is not σ -compact. Then $|P| = 2^{\aleph_0} \kappa(G)$ follows for every G . It remains to be proved that $(Ep) \cap (Ep') = \emptyset$ for $p, p' \in P_0$ with $p \neq p'$.

Take $e, e' \in E$ and $p, p' \in P_0$, $p \neq p'$. By the definition of E and P_0 , we have $e, e' \in E_n \subseteq C_n$ for some n , and $p \in K(d)$, $p' \in K(d')$ for some distinct subsets d, d'

of \mathbb{N} . Let $j \geq n$ be such that $d \cap j \neq d' \cap j$. Since $p \in (d \cap j)_* U_{j+1}$, $p' \in (d' \cap j)_* U_{j+1}$ and $e, e' \in E_n \subseteq C_j$, we get from (3°), $ep \neq e'p'$. \square

(For $\kappa(G) > 2^{\aleph_0}$, the inductive construction is in fact redundant [if one does not need that E be perfect], just take E to be the support of μ , $P_0 = \{e_G\}$ and P_1, P as above).

6. PROOF OF THE MAIN THEOREM — METRIZABLE CASE

Lemma 16. *Let G be a locally compact group. Then we have*

$$Z_t(\mathbf{M}(G)^{**}) \cap \mathbf{M}_a(G)^{**} \subseteq \mathbf{M}(G).$$

Proof. Since $\mathbf{M}_a(G)$ is a subalgebra of $\mathbf{M}(G)$, it follows by elementary arguments that $Z_t(\mathbf{M}(G)^{**}) \cap \mathbf{M}_a(G)^{**} \subseteq Z_t(\mathbf{M}_a(G)^{**})$. Now one can apply the theorem, due to Lau and Losert [LL, Theorem 1] that $Z_t(\mathbf{M}_a(G)^{**}) = \mathbf{M}_a(G)$. \square

Theorem 17. *Let G be a locally compact group, K a compact subgroup such that $|G/K| \leq 2^{\aleph_0} \kappa(G)$. Then $Z_t(\mathbf{M}(G)^{**}) \cap \mathbf{M}(G/K)^{**} \subseteq \mathbf{M}(G/K)$.*

As mentioned in Section 2, the assumption on G/K is satisfied if G/K is metrizable. The case of the trivial subgroup $K = \{e_G\}$ gives the Main Theorem, i.e., $Z_t(\mathbf{M}(G)^{**}) = \mathbf{M}(G)$ holds when $|G| \leq 2^{\aleph_0} \kappa(G)$ (in particular for metrizable groups).

Proof. Put $M_0 = \mathbf{M}_s(G, K)$, $M_1 = \mathbf{M}_a(G, K)$, $M_2 = \mathbf{M}(G/K)$. If $K = \{e_G\}$, we can (using Theorem 15) apply Corollary 11 and get that $Z_t(\mathbf{M}(G)^{**}) \subseteq \mathbf{M}_s(G) \oplus \mathbf{M}_a(G)^{**}$. Since $\mathbf{M}(G) \subseteq Z_t(\mathbf{M}(G)^{**})$, the conclusion now follows from Lemma 16.

In the general case, we take $\widetilde{M}_0 = \mathbf{M}_s(G)$, $\widetilde{M}_1 = \mathbf{M}_a(G)$, $\widetilde{M}_2 = \mathbf{M}(G)$ and apply now Theorem 12. \square

If G is any locally compact group and G_d denotes the same group with discrete topology, then Theorem 17 applies to $G_1 = G \times G_d$ which contains G as an open subgroup. Unfortunately, we do not have a direct argument that strong Arens irregularity of the measure algebra carries over to open subgroups. If one assumes that $2^{\aleph_1} = 2^{\aleph_0}$ (which is consistent with standard set theory), then Theorem 17 applies to groups $G = \prod_{i \in I} G_i$ with G_i compact metrizable and $|I| \leq \aleph_1$, giving examples of compact non-metrizable groups. However, the proof of the full conjecture requires

considerably more work. Theorem 17 (for G/K metrizable) will provide the starting point of an inductive argument.

7. COMPACT SUBGROUPS AND SOME CLASSES OF MEASURES

First, we will give now some formulas for the character of quotient spaces. Then we define some classes of compact subgroups in a non-metrizable group, corresponding classes of measures and decompositions of $\mathbf{M}(G)$. This contains the strongly singular measures mentioned at the beginning.

Lemma 18. *Let G be a locally compact group.*

- (1) *If H_0 and H_1 are closed subgroups of G , $H_0 \supseteq H_1$, then*

$$\chi(G/H_1) = \max(\chi(G/H_0), \chi(H_0/H_1)) .$$

- (2) *If H_i are closed subgroups of G for $i \in I$, then*

$$\chi(G/\bigcap_{i \in I} H_i) \leq \sup_{i \in I} (\chi(G/H_i)) + |I| .$$

- (3) *If G is σ -compact, H a closed subgroup, $N = \bigcap_{x \in G} xHx^{-1}$, then*

$$\chi(G/N) \leq \chi(G/H) + \aleph_0 .$$

In (3), without σ -compactness one has $\chi(G/N) \leq \chi(G/H) + \kappa(G)$.

Proof. Recall that in a Hausdorff space Ω a family $(U_j)_{j \in J}$ of compact neighbourhoods of $\omega \in \Omega$ is a neighbourhood base iff it is downwards directed and satisfies $\bigcap_{j \in J} U_j = \{\omega\}$. Thus, if Ω is locally compact, $\chi(\omega, \Omega)$ is the minimal cardinality of a family of ω -neighbourhoods whose intersection is $\{\omega\}$.

To prove (1), take a family \mathcal{U}_0 of cardinality $\chi(G/H_0)$ consisting of open sets in G with $\bigcap \mathcal{U}_0 = H_0$ and a family \mathcal{U}_1 of cardinality $\chi(H_0/H_1)$ consisting of open sets in G with $\bigcap \mathcal{U}_1 \cap H_0 = H_1$. In addition, we assume that $U = UH_i$ for $U \in \mathcal{U}_i$. Then the image of $\mathcal{U}_0 \cup \mathcal{U}_1$ under the quotient map to G/H_1 has $\{H_1\}$ as the intersection. Conversely, starting with a neighbourhood base in G/H_1 , let \mathcal{U} be a family of open sets in G that is downwards directed and whose intersection is H_1 . We may assume that there is a compact subset C of G such that $U = UH_1 \subseteq CH_1$ for all $U \in \mathcal{U}$. Then one can show that $\{UH_0 : U \in \mathcal{U}\}$ (resp. $\{U \cap H_0 : U \in \mathcal{U}\}$) has intersection H_0 (resp. H_1). Similarly for (2).

To show (3), we can (replacing G by G/N) assume that N is trivial. Let $(V_i)_{i \in I}$ be a family of compact e_G -neighbourhoods such that $\bigcap_{i \in I} V_i \subseteq H$ and $|I| = \chi(G/H)$.

Choose compact symmetric e_G -neighbourhoods W_i such that $W_i^3 \subseteq V_i$ and countable subsets D_i in G such that $G = D_i W_i$. Then the family of sets $\{x W_i x^{-1} : x \in D_i, i \in I\}$ has intersection $\{e_G\}$ and its cardinality is at most $\chi(G/H) + \aleph_0$. Alternatively, when H is compact, one can prove (3) by using from [H], Lemma 2 and formulas (\dagger) and (\ddagger) (which extends to quotient spaces). \square

Corollary 19. *Let $\gamma > 0$ be an ordinal number. If K_α are compact subgroups of G for $\alpha \leq \gamma$, such that $K_\alpha \supseteq K_{\alpha+1}$ and $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$ for all $\alpha < \gamma$, and $K_\beta = \bigcap_{\alpha < \beta} K_\alpha$ for all limit ordinals $\beta \leq \gamma$, then $\chi(G/K_\gamma) = |\gamma| + \chi(G/K_0) + \aleph_0$.*

Proof. By induction, Lemma 18 gives $\chi(G/K_\gamma) \leq |\gamma| + \chi(G/K_0) + \aleph_0$. In addition, equality follows from Lemma 18(1) when $|\gamma| \leq \chi(G/K_0) + \aleph_0$ (since $\gamma > 0$ and $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$). Now assume that equality does not hold for some γ and choose γ minimal. Put $\tau = \chi(G/K_\gamma)$. Then $\tau < |\gamma|$ and $|\alpha| = \chi(G/K_\alpha) \leq \tau$ if $\chi(G/K_0) + \aleph_0 \leq |\alpha|$ and $\alpha < \gamma$. Thus γ must be a limit ordinal. Since $K_\gamma = \bigcap_{\alpha < \gamma} K_\alpha$, it follows that the family of sets $V K_\alpha$ where V is an e_G -neighbourhood and $\alpha < \gamma$ has the intersection K_γ . Thus (see the beginning of the proof of Lemma 18) it defines a neighbourhood basis in G/K_γ (i.e., G/K_γ is homeomorphic to the projective limit of the spaces G/K_α where $\alpha < \gamma$). This implies that there exists a family $(V_i)_{i \in I}$ of e_G -neighbourhoods and $\alpha_i < \gamma$ such that $\bigcap_{i \in I} V_i K_{\alpha_i} = K_\gamma$, where $|I| = \tau$. Put $\beta = \sup\{\alpha_i : i \in I\}$. Since $|\alpha_i| \leq \tau$ for all i and $|I| = \tau$, it follows that $|\beta| \leq \tau$, giving $\beta < \gamma$. But then we would have $V_i K_{\alpha_i} \supset K_\beta$ for all i which is impossible. \square

Now assume that G is a non-discrete locally compact group (i.e., $\chi(G) \geq \aleph_0$; but our main interest will be the case that G is a non-metrizable). Let τ be a cardinal number with $\aleph_0 \leq \tau \leq \chi(G)$. We put

$$\begin{aligned} \mathcal{K}_\tau &= \{K : K \text{ compact subgroup of } G, \chi(G/K) \leq \tau\} \\ \mathcal{K}_\tau^\circ &= \{K : K \text{ compact subgroup of } G, \chi(G/K) < \tau\}. \end{aligned}$$

Corollary 20. *Let G be a locally compact group, $\aleph_0 \leq \tau \leq \chi(G)$. Assume that $K_i \in \mathcal{K}_\tau^\circ$ ($i = 1, \dots, n$) and that H is an open σ -compact subgroup of G . Then it follows $\bigcap_{i=1}^n K_i \in \mathcal{K}_\tau^\circ$, $H \cap K_i \in \mathcal{K}_\tau^\circ$, and there exists $N \in \mathcal{K}_\tau^\circ$ such that N is a normal subgroup of H and $N \subseteq \bigcap_{i=1}^n K_i$.*

Similar properties hold for \mathcal{K}_τ . In fact, $\bigcap_{i \in I} K_i \in \mathcal{K}_\tau$ holds when $K_i \in \mathcal{K}_\tau$ for $i \in I$ and $|I| \leq \tau$. Observe that $\mathcal{K}_\tau = \mathcal{K}_{\tau^+}^\circ$, where τ^+ denotes the successor cardinal

of τ . Furthermore, if H is any open subgroup of a topological group G , then

$$\chi(G/H_1) = \chi(H/(H_1 \cap H)) = \chi(G/(H_1 \cap H))$$

for every subgroup H_1 of G . Every $\mu \in \mathbf{M}(G)$ is supported by some open σ -compact subgroup (depending on μ). This will allow us to reduce many arguments to σ -compact groups (see Lemma 21).

We will now use these classes of compact subgroups to define some classes of measures that will be useful in the induction.

$$\mathbf{M}_\tau(G) = \bigcup_{K \in \mathcal{K}_\tau} \mathbf{M}(G/K) \quad \text{measures of character } \tau.$$

For $\tau = \aleph_0$ we put $\mathbf{M}_{\text{ss}, \aleph_0}(G) = \mathbf{M}_s(G) \cap \mathbf{M}_{\aleph_0}(G)$ and $\mathbf{M}_{\text{ai}, \aleph_0}(G) = \mathbf{M}_a(G)$, and for $\aleph_0 < \tau \leq \chi(G)$

$$\mathbf{M}_{\text{ss}, \tau}(G) = \{ \mu \in \mathbf{M}_s(G) \cap \mathbf{M}_\tau(G) : \mu \perp \mathbf{M}_{\tau_1}(G) \text{ for all } \tau_1 < \tau \}$$

strongly singular measures of character τ

$$\mathbf{M}_{\text{ai}, \tau}(G) = \{ \mu \in \mathbf{M}_\tau(G) : \mu = \lim_{K \in \mathcal{K}_\tau^\circ} \mu \star \lambda_K \text{ (norm limit of the net)} \}$$

approximately invariant measures of character τ

(by Corollary 20, \mathcal{K}_τ° is downwards directed under inclusion).

More generally, when K is a compact subgroup of G that is not open (equivalently, $\chi(G/K) \geq \aleph_0$), we put $\mathbf{M}_{\text{ss}}(G, K) = \mathbf{M}(G/K) \cap \mathbf{M}_{\text{ss}, \chi(G/K)}(G)$ and $\mathbf{M}_{\text{ai}}(G, K) = \mathbf{M}(G/K) \cap \mathbf{M}_{\text{ai}, \chi(G/K)}(G)$. The definitions of $\mathbf{M}_{\text{ss}}(G, K)$ and $\mathbf{M}_{\text{ai}}(G, K)$ impose conditions coming from the bigger space $\mathbf{M}(G)$, using the embedding of $\mathbf{M}(G/K)$ into $\mathbf{M}(G)$ described in Lemma 1 (see also Lemma 33 for a more intrinsic description in terms of G/K). If K is normal in G , things are easier, see Corollary 23 below.

Note that if $\tau_1 < \tau$, then $\mathcal{K}_{\tau_1} \subseteq \mathcal{K}_\tau$ and $\mathbf{M}_{\tau_1}(G) \subseteq \mathbf{M}_\tau(G)$.

Lemma 21. *Let G be a locally compact group, K a non-open compact subgroup.*

- (i) *If H is an open subgroup of G satisfying $K \subseteq H$, then $\mathbf{M}_{\text{ai}}(H, K) = \mathbf{M}_{\text{ai}}(G, K) \cap \mathbf{M}(H)$ and $\mathbf{M}_{\text{ss}}(H, K) = \mathbf{M}_{\text{ss}}(G, K) \cap \mathbf{M}(H)$.*
- (ii) *If K' is a compact subgroup of K with $\chi(G/K') = \chi(G/K)$, then $\mathbf{M}_{\text{ai}}(G, K) = \mathbf{M}_{\text{ai}}(G, K') \cap \mathbf{M}(G/K)$ and $\mathbf{M}_{\text{ss}}(G, K) = \mathbf{M}_{\text{ss}}(G, K') \cap \mathbf{M}(G/K)$.*

Proof. For a compact subgroup K_1 of G , Corollary 20 gives $K_1 \cap H \in \mathcal{K}_\tau$ iff $K_1 \in \mathcal{K}_\tau$, and by Lemma 1, $\mathbf{M}(G/K_1) \cap \mathbf{M}(H) \subseteq \mathbf{M}(H/(K_1 \cap H))$. It follows

that $\mathbf{M}_\tau(G) \cap \mathbf{M}(H) = \mathbf{M}_\tau(H)$, leading to corresponding formulas for $\mathbf{M}_{\text{ss},\tau}$ and $\mathbf{M}_{\text{ai},\tau}$. This implies (i) (alternatively, one might use Lemma 33). (ii) follows immediately from the definitions. \square

Theorem 22. *Let G be a non-discrete locally compact group, τ a cardinal number with $\aleph_0 \leq \tau \leq \chi(G)$, K a non-open compact subgroup of G .*

- (i) $\mathbf{M}_\tau(G)$ and $\mathbf{M}_{\text{ai},\tau}(G)$ are norm closed ideals in $\mathbf{M}(G)$, and $\mathbf{M}_{\text{ss},\tau}(G)$ is a norm closed subspace.
- (ii) $\mathbf{M}_\tau(G)$, $\mathbf{M}_{\text{ai},\tau}(G)$ and $\mathbf{M}_{\text{ss},\tau}(G)$ are L -subspaces (i.e., $\mu \in M$ and $|\nu| \ll |\mu|$ implies $\nu \in M$). We have $\mathbf{M}_\tau(G) = \mathbf{M}_{\text{ss},\tau}(G) \oplus \mathbf{M}_{\text{ai},\tau}(G)$ and $\mathbf{M}_{\text{ss},\tau}(G) \perp \mathbf{M}_{\text{ai},\tau}(G)$.
- (iii) $\mathbf{M}(G/K)$, $\mathbf{M}_{\text{ai}}(G, K)$ and $\mathbf{M}_{\text{ss}}(G, K)$ are closed subspaces and vector sublattices in $\mathbf{M}(G)$ (i.e., $\mu \in M$ implies $|\mu| \in M$). We have

$$\mathbf{M}(G/K) = \mathbf{M}_{\text{ss}}(G, K) \oplus \mathbf{M}_{\text{ai}}(G, K) \quad \text{and} \quad \mathbf{M}_{\text{ss}}(G, K) \perp \mathbf{M}_{\text{ai}}(G, K).$$

Proof. $\mathbf{M}(G/K)$ is always a closed subspace of $\mathbf{M}(G)$ and a left ideal (see Lemma 1). Using (2) of Lemma 18, it follows that $\mathbf{M}_\tau(G)$ is a closed subspace and then the same for $\mathbf{M}_{\text{ai},\tau}(G)$ and $\mathbf{M}_{\text{ss},\tau}(G)$. This implies (looking at the definitions) that $\mathbf{M}_\tau(G)$ and $\mathbf{M}_{\text{ai},\tau}(G)$ are left ideals. Now take $\mu, \nu \in \mathbf{M}(G)$ and let H be an open σ -compact subgroup containing the support of ν . If $\mu \in \mathbf{M}_\tau(G)$, then by (2),(3) of Lemma 18, there exists a normal subgroup N of H such that $N \in \mathcal{K}_\tau$ and $\mu \in \mathbf{M}(G/N)$. As mentioned right after Lemma 1, normality of N implies that $\mathbf{M}(G/N)$ is right H -invariant. It follows that $\mu \star \nu \in \mathbf{M}(G/N) \subseteq \mathbf{M}_\tau(G)$. Thus $\mathbf{M}_\tau(G)$ is a right ideal and a similar argument works for $\mathbf{M}_{\text{ai},\tau}(G)$ (if H is an open σ -compact subgroup of G , the normal subgroups in H define a subset of \mathcal{K}_τ° that is cofinal by Lemma 18 (3)). This proves (i).

To show that $\mathbf{M}_\tau(G)$ is an L -subspace, it will be enough (by closedness) to prove that $h\mu \in \mathbf{M}_\tau(G)$ whenever $\mu \in \mathbf{M}_\tau(G)$ and h is a continuous function of compact support. As above let H be an open σ -compact subgroup such that h vanishes outside H . By the Kakutani-Kodaira theorem (see Lemma 24 below), there exists a compact normal subgroup N of H such that H/N is metrizable and h is N -periodic. If $\mu \in \mathbf{M}(G/K)$ with $K \in \mathcal{K}_\tau$, we have $h\mu \in \mathbf{M}(G/(K \cap N))$ and by (2) of Lemma 18, $K \cap N \in \mathcal{K}_\tau$. It is easy to see that this also implies that $\mathbf{M}_{\text{ss},\tau}(G)$ is an L -subspace. Similar arguments work for $\mathbf{M}_{\text{ai},\tau}(G)$ and show also that $\mathbf{M}_{\text{a}}(G) \subseteq \mathbf{M}_{\aleph_0}(G)$. $\mathbf{M}_{\text{ss},\tau}(G) \perp \mathbf{M}_{\text{ai},\tau}(G)$ follows from the definition. Observe that $\mathbf{M}_{\tau_1}(G) \subseteq \mathbf{M}_{\text{ai},\tau}(G)$ holds for $\tau_1 < \tau$. Hence if $\mu \in \mathbf{M}_\tau(G)$ and $\mu \perp \mathbf{M}_{\text{ai},\tau}(G)$ then

$\mu \in \mathbf{M}_s(G)$ and combined $\mu \in \mathbf{M}_{ss,\tau}(G)$. Now take any $\mu \in \mathbf{M}_\tau(G)$ with $\mu \geq 0$. If there exists $\nu \in \mathbf{M}_{ai,\tau}(G)$ such that $\nu \neq 0$ and ν is not singular to μ , we can (decomposing and taking advantage of the properties of an L-subspace) assume that $0 \leq \nu \leq \mu$. Then (using closedness and again lattice properties) we can find such a ν with $\|\nu\|$ maximal and it follows that $\mu - \nu \perp \mathbf{M}_{ai,\tau}(G)$. As mentioned above, this implies $\mu - \nu \in \mathbf{M}_{ss,\tau}(G)$. It follows that $\mathbf{M}_\tau(G) = \mathbf{M}_{ss,\tau}(G) \oplus \mathbf{M}_{ai,\tau}(G)$ (for $\tau = \aleph_0$ one gets just the Lebesgue decomposition of elements of $\mathbf{M}_{\aleph_0}(G)$ with respect to λ_G).

The statements in (iii) about $\mathbf{M}(G/K)$ and its subspaces are easy consequences of (i) and (ii). \square

Corollary 23. *If K is a compact normal subgroup of G that is not open, then the isomorphism of Lemma 1 maps the subspace $\mathbf{M}_{ss}(G, K)$ onto $\mathbf{M}_{ss,\chi(G/K)}(G/K)$ and $\mathbf{M}_{ai}(G, K)$ onto $\mathbf{M}_{ai,\chi(G/K)}(G/K)$.*

Proof. Let $\Phi(\mu) = \dot{\mu}$ be this isomorphism ($\mu \in \mathbf{M}(G, K)$). As mentioned earlier (the image measure of Haar measure is Haar measure), $\Phi(\mathbf{M}_s(G) \cap \mathbf{M}(G, K)) = \mathbf{M}_s(G/K)$. For a compact subgroup $L \supseteq K$, we have $\Phi(\mathbf{M}(G, L)) = \mathbf{M}(G/K, L/K)$ and it follows that $\Phi^{-1}(\mathbf{M}_\tau(G/K)) \subseteq \mathbf{M}_\tau(G)$ holds for $\aleph_0 \leq \tau \leq \chi(G/K)$. Φ being order preserving, this implies that $\Phi^{-1}(\mathbf{M}_{ss,\chi(G/K)}(G/K)) \supseteq \mathbf{M}_{ss}(G, K)$ and similarly $\Phi^{-1}(\mathbf{M}_{ai,\chi(G/K)}(G/K)) \supseteq \mathbf{M}_{ai}(G, K)$. But then by (ii), (iii) of Theorem 22, these inclusions must be equalities. \square

Remarks. From (ii) of the last theorem, the following decomposition results: Every $\mu \in \mathbf{M}_s(G)$ has a unique (up to reorderings) representation $\mu = \sum \mu_i$, where $\mu_i \in \mathbf{M}_{ss,\tau_i}(G)$ for some pairwise different cardinals τ_i with $\aleph_0 \leq \tau_i \leq \chi(G)$. Thus $\mathbf{M}_s(G)$ is the l^1 -sum of all the spaces $\mathbf{M}_{ss,\tau}(G)$ with $\aleph_0 \leq \tau \leq \chi(G)$. Similarly, for $\aleph_0 \leq \tau \leq \chi(G)$, $\mathbf{M}_{ai,\tau}(G)$ is the l^1 -sum of $\mathbf{M}_a(G)$ and all the spaces $\mathbf{M}_{ss,\tau'}(G)$ with $\aleph_0 \leq \tau' < \tau$.

As mentioned in the proof of (ii) above, $\mathbf{M}_a(G) \subseteq \mathbf{M}_{\aleph_0}(G)$ holds. It follows that in the definition of $\mathbf{M}_{ss,\tau}(G)$, the condition $\mu \in \mathbf{M}_s(G)$ is redundant for $\tau > \aleph_0$.

For a totally disconnected group G one can still show that $\mu = \lim_{K \in \mathcal{K}_{\aleph_0}^\circ} \mu \star \lambda_K$ for $\mu \in \mathbf{M}_a(G)$. But for other groups G this is no longer true, since $\mathcal{K}_{\aleph_0}^\circ$ (these are just the open compact subgroups) is not sufficiently large. It may even happen (e.g. for $G = \mathbb{R}$) that $\mathcal{K}_{\aleph_0}^\circ$ is empty.

8. SEPARATION OF STRONGLY SINGULAR MEASURES

In this section we extend Theorem 15 to the case of strongly singular measures (Theorem 30). This will be used in the proof of the general case of the Main Theorem. Recall the Kakutani-Kodaira theorem (see [H] Theorem 3 for a more general version, a more special case—compactly generated groups—which would be sufficient for our purpose (when modifying slightly some arguments) is in [HR, 8.7]).

Lemma 24. *Let H be a locally compact, σ -compact group. If U_n , $n < \omega$, are neighbourhoods of e_H then there exists a compact normal subgroup N of H such that $N \subseteq U_n$ for all n and H/N is metrizable.*

Lemma 25. *If G is any locally compact group then G has a compact subgroup K_G such that G/K_G is metrizable. If G is non-metrizable, then for any such group it follows that $\chi(K_G) = \chi(G)$. If G is σ -compact, K_G can be chosen to be normal.*

Proof. Take any symmetric compact neighbourhood U of e_G . Then $G_0 = \bigcup_{n=1}^{\infty} U^n$ is an open subgroup of G (cf. [HR, 5.7]), hence $\chi(G_0) = \chi(G)$. Since G_0 is compactly generated, by Lemma 24 there is a compact normal subgroup K_G of G_0 such that G_0/K_G is metrizable and thus G/K_G is metrizable.

If $\chi(G) > \aleph_0$, then $\chi(K_G) = \chi(G)$ since $\chi(G) = \max(\chi(G/K_G), \chi(K_G))$ by Lemma 18. \square

Lemma 26. *Let G be a σ -compact, locally compact group and $\nu, \nu' \in \mathbf{M}(G)$, $\nu, \nu' \geq 0$. If $\nu \perp \nu'$ then there exist a \mathbf{K}_σ -set $E \subseteq G$ and a compact normal subgroup N of G such that $\nu(E) = \nu(G)$, $\nu'(EN) = 0$ and $\chi(G/N) \leq \aleph_0$. In particular, $\nu \star \lambda_N \perp \nu' \star \lambda_N$.*

If V is an e_G -neighbourhood in G , we can find such an N with $N \subseteq V$.

Proof. Since $\nu \perp \nu'$, there is a \mathbf{K}_σ -set $E \subseteq G$ such that $\nu(E) = \nu(G)$ and $\nu'(E) = 0$. Thus there are compact sets C_n such that $E = \bigcup_{n=1}^{\infty} C_n$. For every n and m there exists an open set $W_{n,m} \supseteq C_n$ satisfying $\nu'(W_{n,m}) < 2^{-m}$. Since C_n are compact, there are neighbourhoods $U_{n,m}$ of e_G with $C_n U_{n,m} \subseteq W_{n,m}$. By Lemma 24 there is a compact normal subgroup N of G such that $N \subseteq \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} U_{n,m} \cap V$ and $\chi(G/N) \leq \aleph_0$. Then $\nu'(C_n N) \leq \nu'(C_n U_{n,m}) \leq \nu'(W_{n,m}) < 2^{-m}$ for every m , giving $\nu'(C_n N) = 0$ for every n . Since $EN = \bigcup_{n=1}^{\infty} C_n N$, it follows that $\nu'(EN) = 0$. Thus ν and $\nu \star \lambda_N$ are concentrated on EN , whereas $\nu' \star \lambda_N(EN) = \nu'(EN) = 0$. \square

Lemma 27. *Let G be a σ -compact, locally compact group such that $\chi(G) > \aleph_0$, K_1 a normal subgroup of G with $K_1 \in \mathcal{K}_{\chi(G)}^\circ$, V an e_G -neighbourhood, and let $\mu \in \mathbf{M}_{\text{ss}, \chi(G)}(G)$ with $\mu \geq 0$, $\mu(G) = 1$. Then there exists a normal subgroup K_2 of G with $K_2 \in \mathcal{K}_{\chi(G)}^\circ$ and a \mathbf{K}_σ -set $E \subseteq G$ such that $K_2 \subseteq K_1 \cap V$, $\mu(E) = 1$, $\mu \star \lambda_{K_1}(EK_2) = 0$ and K_1/K_2 is metrizable.*

Proof. Since $K_1 \in \mathcal{K}_{\chi(G)}^\circ$, we have $\mu \perp \mu \star \lambda_{K_1}$. Let N be as in Lemma 26 with $\nu = \mu$ and $\nu' = \mu \star \lambda_{K_1}$. Put $K_2 = K_1 \cap N$. Recall ([HR] Thm. 5.33) that K_1/K_2 is topologically isomorphic to K_1N/N . Since $\chi(G/N) \leq \aleph_0$, we get that K_1/K_2 is metrizable and by (1) of Lemma 18 that $K_2 \in \mathcal{K}_{\chi(G)}^\circ$. \square

Corollary 28. *For K_1, K_2, μ as above, the following properties hold:*

- (1) $\mu \star \lambda_{K_1} \perp \mu \star \lambda_{K_2}$,
- (2) $\mu \star K_2 \perp \mu \star xK_2$ for λ_{K_1} -almost all $x \in K_1$,
- (3) $\chi(K_1/K_2) = \aleph_0$ and $K_2 \in \mathcal{K}_{\chi(G)}^\circ$.

We use here the notations from the beginning of Section 4.

Proof. We have

$$\int_{K_1} \mu \star x(EK_2) d\lambda_{K_1}(x) = \int_{K_1} \mu(EK_2x^{-1}) d\lambda_{K_1}(x) = \mu \star \lambda_{K_1}(EK_2) = 0$$

and therefore $\mu \star x(EK_2) = 0$ for λ_{K_1} -almost all $x \in K_1$.

Take any $x \in K_1$ such that $\mu \star x(EK_2) = 0$ and any $y, y' \in K_2$. Then

$$\mu \star xy(Ey') = \mu \star x \star y(Ey') = \mu \star x(Ey'y^{-1}) \leq \mu \star x(EK_2) = 0,$$

while $\mu \star y'(Ey') = \mu(E) = 1$ which shows that $\mu \star y' \perp \mu \star xy$. Thus $\mu \star K_2 \perp \mu \star xK_2$. If K_2 were open in K_1 , then $\lambda_{K_1}(K_2) > 0$ and we would get $\mu \star K_2 \perp \mu \star K_2$. Thus K_1/K_2 must be non-discrete, giving $\chi(K_1/K_2) = \aleph_0$. \square

As usual, a cardinal number τ will be identified with the minimal ordinal number α for which $|\alpha| = \tau$.

Lemma 29. *Let G be a σ -compact, locally compact group, K_0 a compact normal subgroup such that $\chi(G) > \chi(G/K_0) + \aleph_0$ and $\mu \in \mathbf{M}_{\text{ss}, \chi(G)}(G)$ with $\mu \geq 0$, $\mu(G) = 1$. Then for $0 < \alpha \leq \chi(G)$ there exist compact normal subgroups K_α of G and $x_\alpha \in K_\alpha$ such that*

- (i) $K_\alpha \supseteq K_{\alpha+1}$ and $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$ for all $\alpha < \chi(G)$;
- (ii) $K_\beta = \bigcap_{\alpha < \beta} K_\alpha$ for all limit ordinals $\beta \leq \chi(G)$;

- (iii) $K_{\chi(G)} = \{e_G\}$;
- (iv) $\mu \star K_{\alpha+1} \perp \mu \star x_\alpha K_{\alpha+1}$ and $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_{\alpha+1}}$ for all $\alpha < \chi(G)$.

Note that by Corollary 19 it follows that $\chi(G/K_\alpha) = |\alpha| + \chi(G/K_0) + \aleph_0$ for $0 < \alpha \leq \chi(G)$, in particular $K_\alpha \in \mathcal{K}_{\chi(G)}^\circ$ for $\alpha < \chi(G)$, and K_α is not open in G for $\alpha > 0$.

Proof. Let $(V_\alpha)_{\alpha < \chi(G)}$ be a family of e_G -neighbourhoods satisfying $\bigcap_{\alpha < \chi(G)} V_\alpha = \{e_G\}$. Now use transfinite induction, applying Lemma 27 and Corollary 28 with the requirement $K_{\alpha+1} \subseteq V_\alpha$. \square

Theorem 30. *Let G be a locally compact group, $\aleph_0 \leq \tau \leq \chi(G)$ and $\mu \in \mathbf{M}_{\text{ss}, \tau}(G)$. There exists a set $P \subseteq G$ such that $|P| = 2^\tau$ and $\mu \star p \perp \mu \star p'$ for all $p, p' \in P$, $p \neq p'$. Thus μ is 2^τ -thin.*

If G is non-metrizable, we can find such a P satisfying $P \subseteq K_G$ (defined by Lemma 25).

Proof. We have $\mu \in \mathbf{M}_{\text{ss}}(G, K)$ for some compact subgroup K with $\chi(G/K) = \tau$. The case $\tau = \aleph_0$ was settled in Theorem 15 (recall that $\mathbf{M}_{\text{ss}}(G, K) \subseteq \mathbf{M}_{\text{s}}(G)$). Thus, we may assume that $\tau > \aleph_0$ and, replacing G by some open subgroup (using Lemma 21), we may assume that G is σ -compact. Then by (3) of Lemma 18, K can be replaced by a normal subgroup. In this way (using Corollary 23), the proof is reduced to the case where $\mu \in \mathbf{M}_{\text{ss}, \chi(G)}(G)$ and G is σ -compact and non-metrizable (i.e., $\tau = \chi(G) > \aleph_0$). Without loss of generality, assume that $\mu \geq 0$, $\mu(G) = 1$. Recall the notational conventions in the proof of Theorem 15.

Let K_α and $x_\alpha \in K_\alpha$ for $\alpha < \chi(G)$ be as in Lemma 29, with $K_0 = K_G$ (obtained from Lemma 25). Induction on $\beta \leq \chi(G)$ will be used to construct elements $d_* \in K_0$ for all $d \subseteq \beta$ such that

- (1 \bullet) if $\alpha < \beta \leq \chi(G)$ and $d \subseteq \beta$ then $d_* \in (d \cap \alpha)_* K_\alpha$;
- (2 \bullet) if $\beta \leq \chi(G)$ and d and d' are distinct subsets of β then $\mu \star d'_* K_\beta \perp \mu \star d_* K_\beta$.

That will conclude the proof, since in view of (2 \bullet) we can take $P = \{d_* : d \subseteq \chi(G)\}$.

To carry out the induction, start with $\beta = 0$ and define $\emptyset_* = e_G$. Now assume that $0 \neq \beta < \chi(G)$ and that d_* have been defined for all $d \subseteq \alpha$ and $\alpha < \beta$. If $d \subseteq \beta$ and β is a limit ordinal define d_* to be any point in $\bigcap_{\alpha < \beta} (d \cap \alpha)_* K_\alpha$ (this is a decreasing family of sets by (1 \bullet) and since (K_α) is decreasing). If $\beta = \alpha + 1$ then define

$$d_* = \begin{cases} (d \cap \alpha)_* & \text{when } \alpha \notin d \\ x_\alpha(d \cap \alpha)_* & \text{when } \alpha \in d. \end{cases}$$

Slightly informally, this can be described as follows. If $\beta = \gamma + n$, where n is finite, $d \subseteq \beta$ and $\beta \setminus \gamma = \{\gamma_0, \dots, \gamma_l\}$, where $\gamma \leq \gamma_0 < \dots < \gamma_l < \beta$, then $d_* = x_{\gamma_l} \cdots x_{\gamma_0}(d \cap \gamma)_*$ (for technical reasons, we revert here the order of the factors, compared to the proof of Theorem 15). For limit ordinals β there is some choice in the definition of d_* , since the net of “partial products” $((d \cap \alpha)_*)_{\alpha < \beta}$ will converge in general only in the quotient G/K_β (being the projective limit of $(G/K_\alpha)_{\alpha < \beta}$; compare the proof of Corollary 19).

Property (1 \bullet) is easy to prove from the definition, using normality of K_α in G . To prove (2 \bullet), if $d, d' \subseteq \beta$ with $d \neq d'$, take $\beta' \leq \beta$ minimal such that $d \cap \beta' \neq d' \cap \beta'$. In view of (1 \bullet), replacing β, d, d' by $\beta', d \cap \beta', d' \cap \beta'$, it is enough to consider the case where $\beta = \alpha + 1$ and $d, d' \subseteq \beta$ are such that $d' \cap \alpha = d \cap \alpha$, $d_* = (d \cap \alpha)_*$ and $d'_* = x_\alpha(d \cap \alpha)_* = x_\alpha d_*$. Since $\mu \star K_{\alpha+1} \perp \mu \star x_\alpha K_{\alpha+1}$, we have also $\mu \star K_{\alpha+1} d_* \perp \mu \star x_\alpha K_{\alpha+1} d_*$. As $K_{\alpha+1}$ is a normal subgroup of G ,

$$\begin{aligned} K_{\alpha+1} d_* &= d_* K_{\alpha+1} = d_* K_\beta \\ x_\alpha K_{\alpha+1} d_* &= x_\alpha d_* K_{\alpha+1} = d'_* K_\beta. \end{aligned}$$

Thus $\mu \star d_* K_\beta \perp \mu \star d'_* K_\beta$. □

Corollary 31. *Let G be a locally compact group, K a non-open compact subgroup. Then every $\mu \in \mathbf{M}_{\text{ss}}(G, K)$ is $|G/K|$ -thin.*

Proof. Recall that $|G/K| = \kappa(G) \cdot 2^{\chi(G/K)}$ holds when K has infinite index in G . By Theorem 15, μ is $\kappa(G)$ -thin and by Theorem 30 (using that $\mathbf{M}_{\text{ss}}(G, K) \subseteq \mathbf{M}_{\text{ss}, \chi(G/K)}(G)$), μ is $2^{\chi(G/K)}$ -thin when K is not open. This covers all possible cases. □

Remarks. We mention here some further results that can be shown by similar methods (this is not needed for the proof of the Main Theorem).

(a) There are some converse statements to Corollary 31. For $\mu \in \mathbf{M}(G)$, $\mu \neq 0$, put $K_\mu = \{x \in G : \mu \star x = \mu\}$ and $M_\mu = \{\mu \star x : x \in G\}$. Then K_μ is the maximal compact subgroup of G such that $\mu \in \mathbf{M}(G/K_\mu)$. Clearly, we have $d(M_\mu) \leq |M_\mu| = |G/K_\mu|$. In particular, μ cannot be τ -thin for $\tau > |G/K_\mu|$. Hence the value $|G/K|$ in Corollary 31 is the best possible when $\mu \neq 0$.

The equality $d(M_\mu) = |G/K_\mu|$ holds iff either M_μ is finite (e.g. when G is finite) or $\kappa(G) \geq 2^{\chi(G/K_\mu)}$ or $\mu \notin \mathbf{M}_{\text{ai}, \tau}(G)$ where τ denotes the least cardinal such that

$2^\tau = 2^{\chi(G/K_\mu)}$ (if the first two options do not hold then K_μ must be non-open). In the remaining cases, $d(M_\mu)$ can be expressed similarly, using the decomposition of $\mathbf{M}_s(G)$ described at the end of Section 7. For $\mu \notin \mathbf{M}_a(G)$ (which implies that K_μ is not open), μ is $d(M_\mu)$ -thin iff either $\kappa(G) \geq 2^{\chi(G/K_\mu)}$ or $\mu \perp \mathbf{M}_{ai,\tau}(G)$ (with τ as above). In all these cases $d(M_\mu) = |G/K_\mu|$ holds (but not conversely).

Assuming e.g. the generalized continuum hypothesis, it follows that if K is a non-open compact subgroup with $\kappa(G) < 2^{\chi(G/K)}$ and $\mu \in \mathbf{M}(G/K)$ is $|G/K|$ -thin, then μ must be strongly singular. On the other hand, under the assumption $2^{\aleph_1} = 2^{\aleph_0}$ (which is also consistent with ZFC), one gets that for a group G with $\chi(G) \leq \aleph_1$ all $\mu \in \mathbf{M}_s(G)$ are $|G|$ -thin.

(b) Another generalization is to consider a locally compact space Ω with a (jointly continuous) left action of a locally compact group G (see [Jw] and [La]). This induces a left action of $\mathbf{M}(G)$ on $\mathbf{M}(\Omega)$ which will be written again as $\mu \star \nu$ for $\mu \in \mathbf{M}(G)$, $\nu \in \mathbf{M}(\Omega)$. There need not exist a non-zero G -invariant measure on Ω (and if one exists, it need not be unique), but one can define e.g. $\mathbf{M}_a(\Omega) = \mathbf{M}_a(G) \star \mathbf{M}(\Omega)$. If one uses now *left* translations by elements of G , most of the constructions in Sections 5 and 8 work again. For example, every $\mu \in \mathbf{M}_s(\Omega)$ ($= \mathbf{M}_a(\Omega)^\perp$) is 2^{\aleph_0} -thin (with respect to left translations from G). Similarly, one can define strongly singular measures on Ω and there is an analogue of Theorem 30. In these cases one uses translations by elements from some compact subset of G . The statements based on non-compactness (i.e., using $\kappa(G)$) do not always extend to this setting, depending on further properties of the action.

9. PROOF OF THE MAIN THEOREM — GENERAL CASE

First we show some further properties of the spaces $\mathbf{M}_{ss}(G, K)$ and $\mathbf{M}_{ai}(G, K)$. Then (Theorem 40) we arrive at the inductive argument to prove the Main Theorem.

Lemma 32. *Let G be a locally compact group, K a compact subgroup of G , $\mu \in \mathbf{M}(G)$. We have $\mu \perp \mathbf{M}(G/K)$ iff $\mu \perp |\mu| \star \lambda_K$. If $\nu \in \mathbf{M}(G/K)$, $\nu \perp |\mu| \star \lambda_K$, then $\nu \perp \mu$.*

Proof. It is enough to consider $\mu \geq 0$ and $\nu \geq 0$. For clarity, we take up the more precise notation of Lemma 1. For the second statement, assume that $\nu \in \mathbf{M}(G, K)$ and $\nu \perp \mu \star \lambda_K$. Then for the image measures, $\dot{\nu} \perp (\mu \star \lambda_K)^\cdot$ holds as well (by Lemma 1). Hence there is a Borel set E_0 in G/K such that $\dot{\nu}(E_0) = \dot{\nu}(G/K)$ and

$(\mu \star \lambda_K)^\cdot(E_0) = 0$. Recall that $(\mu \star \lambda_K)^\cdot = \dot{\mu}$. Put $E = \pi^{-1}(E_0)$. Then E is a Borel set in G and, by the definition of image measures, we get $\nu(E) = \nu(G)$ and $\mu(E) = 0$, so that $\nu \perp \mu$.

In the first part, one direction is clear. For the other one, assume that $\mu \perp \mu \star \lambda_K$ and take any $\nu \in \mathbf{M}(G, K)$ ($\nu \geq 0$). The Lebesgue decomposition of ν with respect to $\mu \star \lambda_K \in \mathbf{M}(G, K)$ yields $\nu = \nu_0 + \nu_1$ where $\nu_0, \nu_1 \in \mathbf{M}(G, K)$, $\nu_0, \nu_1 \geq 0$, $\nu_0 \perp \mu \star \lambda_K$ and $\nu_1 \ll \mu \star \lambda_K$. The second statement of the lemma implies $\nu_0 \perp \mu$ and from $\nu_1 \ll \mu \star \lambda_K$ and $\mu \perp \mu \star \lambda_K$ we get $\nu_1 \perp \mu$. Hence $\nu \perp \mu$. \square

If G is a locally compact group, K a compact subgroup of G and G/K non-metrizable, put

$$\mathcal{K}_K = \mathcal{K}_K(G) = \{L \supseteq K : L \text{ compact subgroup of } G, \chi(G/L) < \chi(G/K)\}.$$

By (2) in Lemma 18 the family \mathcal{K}_K is downwards directed by inclusion. For $L \in \mathcal{K}_K$, recall that (using the identifications of Lemma 1) $\mathbf{M}(G/L)$ is a subalgebra of $\mathbf{M}(G/K)$. The next Lemma will provide a more intrinsic description of the subspaces $\mathbf{M}_{\text{ai}}(G, K)$ and $\mathbf{M}_{\text{ss}}(G, K)$ of $\mathbf{M}(G/K)$.

Lemma 33. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable, and $\nu \in \mathbf{M}(G)$. We have $\nu \in \mathbf{M}_{\text{ai}}(G, K)$ iff $\nu = \lim_{L \in \mathcal{K}_K} \nu \star \lambda_L$ (norm limit). This is also equivalent to $\inf_{L \in \mathcal{K}_K} \|\nu - \nu \star \lambda_L\| = 0$. For $\nu \in \mathbf{M}(G/K)$ we have $\nu \in \mathbf{M}_{\text{ss}}(G, K)$ iff $\nu \perp |\nu| \star \lambda_L$ for all $L \in \mathcal{K}_K$.*

In particular, $\mathbf{M}_{\text{ai}}(G, K)$ coincides with the norm-closure of $\bigcup_{L \in \mathcal{K}_K} \mathbf{M}(G/L)$. Moreover, it follows easily that $\delta_x \star \lambda_K$ (or more generally, every $\mu \in \mathbf{M}(G/K)$ with $\mu \ll \delta_x \star \lambda_{K'}$ for some compact subgroup K' of G with $\chi(G/K') = \chi(G/K)$) belongs to $\mathbf{M}_{\text{ss}}(G, K)$ for all $x \in G$.

An important step in the proof of the Main Theorem will be to show that the limit condition for \mathbf{M}_{ai} can be weakened (Proposition 39): if the net $(\nu \star \lambda_L)_{L \in \mathcal{K}_K}$ converges in the *weak* topology* of $\mathbf{M}(G)^{**}$ (to any limit), it already follows that $\nu \in \mathbf{M}_{\text{ai}}(G, K)$.

Proof. Put $\tau = \chi(G/K)$. If L, L' are compact subgroups of G with $L \subseteq L'$, then $\lambda_{L'} \star \lambda_L = \lambda_{L'}$. Thus $\|\nu \star \lambda_{L'} - \nu \star \lambda_L\| \leq \|\nu \star \lambda_{L'} - \nu\|$ and it follows that $\|\nu \star \lambda_L - \nu\| \leq 2 \|\nu \star \lambda_{L'} - \nu\|$. This shows that $\nu = \lim_{L \in \mathcal{K}_K} \nu \star \lambda_L$ is equivalent to $\inf_{L \in \mathcal{K}_K} \|\nu - \nu \star \lambda_L\| = 0$ and this equivalence persists when \mathcal{K}_K is replaced by any family \mathcal{D} of compact subgroups which is downwards directed.

Since $\mathcal{K}_K \subseteq \mathcal{K}_\tau^\circ$, it follows that $\nu = \lim_{L \in \mathcal{K}_K} \nu \star \lambda_L$ implies $\nu = \lim_{L \in \mathcal{K}_\tau^\circ} \nu \star \lambda_L$, i.e. $\nu \in \mathbf{M}_{\text{ai},\tau}(G)$. Furthermore, $\nu \star \lambda_L \in \mathbf{M}(G/K)$ for $L \in \mathcal{K}_K$ implies $\nu \in \mathbf{M}(G/K)$.

For the converse, if $L \in \mathcal{K}_\tau^\circ$, let H be an open σ -compact subgroup of G containing K and L . By Corollary 20 there exists $L' \in \mathcal{K}_\tau^\circ$ such that L' is normal in H and $L' \subseteq L$. Then KL' is a group, by (1) of Lemma 18 we have $KL' \in \mathcal{K}_K$ and for $\nu \in \mathbf{M}(G/K)$ we get (using $\lambda_{KL'} = \lambda_K \star \lambda_{L'}$) that $\nu \star \lambda_{KL'} = \nu \star \lambda_{L'}$. Thus $\nu = \lim_{L \in \mathcal{K}_\tau^\circ} \nu \star \lambda_L$ implies $\nu = \lim_{L \in \mathcal{K}_K} \nu \star \lambda_L$.

For the second part, we may assume that $\nu \geq 0$. By Theorem 22, we have $\nu = \nu_1 + \nu_2$, where $\nu_1 \in \mathbf{M}_{\text{ss}}(G, K)$, $\nu_2 \in \mathbf{M}_{\text{ai}}(G, K)$. If $\nu_2 \neq 0$ we have by the first part $\nu_2 \not\perp \nu_2 \star \lambda_L$ for some $L \in \mathcal{K}_K$. Since $0 \leq \nu_2 \leq \nu$, this implies $\nu \not\perp \nu \star \lambda_L$. The other direction follows immediately from the definition of $\mathbf{M}_{\text{ss}}(G, K)$. \square

Corollary 34. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable, and $\nu \in \mathbf{M}(G)$. Let $\mathcal{D} \subseteq \mathcal{K}_{\chi(G/K)}^\circ$ be a family of compact subgroups of G that is cofinal with \mathcal{K}_K (i.e., for $L \in \mathcal{K}_K$ there exists $L' \in \mathcal{D}$ with $L' \subseteq L$). We have $\nu \in \mathbf{M}_{\text{ai}}(G, K)$ iff $\inf_{L \in \mathcal{D}} \|\nu - \nu \star \lambda_L\| = 0$. For $\nu \in \mathbf{M}(G/K)$ we have $\nu \in \mathbf{M}_{\text{ss}}(G, K)$ iff $\nu \perp |\nu| \star \lambda_L$ for all $L \in \mathcal{D}$. Assuming in addition that \mathcal{D} is downwards directed, it follows that $\nu \in \mathbf{M}_{\text{ai}}(G, K)$ iff $\nu = \lim_{L \in \mathcal{D}} \nu \star \lambda_L$ (norm limit).*

Proof. The first part and the final statement follow immediately from Lemma 33 and its proof. For the second part, observe that by Lemma 32, $\nu \perp |\nu| \star \lambda_{L'}$ implies $\nu \perp |\nu| \star \lambda_L$ for all $L \supseteq L'$ (since $\mathbf{M}(G/L) \subseteq \mathbf{M}(G/L')$). \square

Lemma 35. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable. Put $\tau = \chi(G/K)$ and let $\mathcal{D} = \{K_\alpha : \alpha < \tau\}$ be a subfamily of \mathcal{K}_K such that $K_\alpha \supseteq K_\beta$ for $\alpha < \beta < \tau$, and $K = \bigcap_{\alpha < \tau} K_\alpha$. If τ is a successor cardinal (i.e., τ is not the supremum of the family of smaller cardinals), then \mathcal{D} is cofinal in \mathcal{K}_K .*

Proof. The argument is similar as in the proof of Corollary 19. The family of sets VK_α where V is an e_G -neighbourhood and $\alpha < \tau$ defines a neighbourhood basis in G/K . Given $L \in \mathcal{K}_K$, it follows that there exists a family $(V_i)_{i \in I}$ of e_G -neighbourhoods and $\alpha_i < \tau$ such that $\bigcap_{i \in I} V_i K_{\alpha_i} \subseteq L$, where $|I| = \chi(G/L) < \tau$. Then the assumptions about τ imply that $\beta = \sup\{\alpha_i : i \in I\} < \tau$ and monotonicity gives $K_\beta \subseteq L$. \square

Remark. The same proof works for limit cardinals τ that are regular (i.e., if τ cannot be expressed as the supremum of a set of cardinality less than τ , whose elements are cardinals less than τ). Note that in these cases, it follows by Lemma 33 that $\nu = \lim_{\alpha < \tau} \nu \star \lambda_{K_\alpha}$ (norm limit) holds for all $\nu \in \mathbf{M}_{\mathbf{ai}}(G, K)$ and (K_α) as above. One might expect this to be valid without restrictions on τ , but the following example shows a different behaviour. Let F be a non-trivial finite group and consider a product group $G = F^\tau$ for an infinite cardinal τ and $K = \{e\}$. For infinite $\alpha < \tau$ take $K_\alpha = F^{\tau \setminus \alpha}$ (embedded into G in the usual way). For a non-empty subset I of τ with $\aleph_0 \leq |I| < \tau$ take $L = F^{\tau \setminus I}$. Then $G/K_\alpha \cong F^\alpha$, $G/L \cong F^I$, $\chi(G/K) = \chi(G) = \tau$, $\chi(G/K_\alpha) = |\alpha|$, $\chi(G/L) = |I|$. Thus $K_\alpha, L \in \mathcal{K}_K$ and clearly $\bigcap_\alpha K_\alpha = K$. If I can be chosen so that $\sup I = \tau$ (this works for any limit cardinal that is not regular), then $K_\alpha \not\subseteq L$ for all α , hence $\{K_\alpha\}$ is not cofinal in \mathcal{K}_K . Furthermore, taking $\nu = \lambda_L \in \mathbf{M}_{\mathbf{ai}}(G, K) = \mathbf{M}_{\mathbf{ai}, \tau}(G)$, it is easy to see that $\nu \star \lambda_{K_\alpha} = \lambda_{LK_\alpha} \perp \nu$ for all α (since $I \setminus \alpha$ must be infinite), hence $(\nu \star \lambda_{K_\alpha})$ does not converge to ν in norm (but, in fact one can see as in Corollary 37 (2) below that no other limit is possible).

Lemma 36. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable, $L \in \mathcal{K}_K$ and $\mu \in \mathbf{M}_{\mathbf{ss}}(G, K)$. Put $\tau = \chi(G/K)$. There exists a family $\mathcal{D} = \{K_\alpha : \alpha < \tau\} \subseteq \mathcal{K}_K$ such that $L \supseteq K_\alpha \supseteq K_\beta$ and $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_\beta}$ for $\alpha < \beta < \tau$, $K_\beta = \bigcap_{\alpha < \beta} K_\alpha$ for all limit ordinals $\beta < \tau$, $\chi(G/K_\alpha) = \chi(G/L) + |\alpha|$ when $\alpha < \tau$ is infinite, and $K = \bigcap_{\alpha < \tau} K_\alpha$.*

Observe that in Lemma 25 the group K_G can always be chosen to be normalized by a given compact subgroup K of G . Then $L = K_G K$ can be used for Lemma 36 and then the corresponding family \mathcal{D} satisfies $\chi(G/K_\alpha) = |\alpha|$ when α is infinite.

Proof. Consider an open σ -compact subgroup $H \supseteq L$ that contains the support of μ . By Corollary 20 and Lemma 21, we may replace G by H and assume that G is σ -compact. Replacing μ by $|\mu|$, we assume $\mu \geq 0$ (note that if $|\mu| \star \lambda_{K_\alpha} \perp |\mu| \star \lambda_{K_\beta}$ then $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_\beta}$ and use Theorem 22). Adding λ_K if necessary, we can assume $\mu \neq 0$.

Put $N = \bigcap_{x \in G} xKx^{-1}$, $N_L = \bigcap_{x \in G} xLx^{-1}$, then $N \subseteq K \cap L$, both N, N_L are normal in G , $\chi(G/N) = \tau$, $\chi(G/N_L) = \chi(G/L)$ (Lemma 18) and $\mu \in \mathbf{M}_{\mathbf{ss}}(G, N)$ (Lemma 21). Corollary 23 gives $\mu \in \mathbf{M}_{\mathbf{ss}, \tau}(G/N)$. We put $\tilde{K}_0 = N_L/N$ and apply Lemma 29 to $\tilde{G} = G/N$ and μ . This produces a family of compact normal subgroups \tilde{K}_α of \tilde{G} ($\alpha < \tau$) such that $\tilde{K}_\alpha \supseteq \tilde{K}_{\alpha+1}$, $\chi(\tilde{K}_\alpha/\tilde{K}_{\alpha+1}) = \aleph_0$ and $\mu \star \lambda_{\tilde{K}_\alpha} \perp \mu \star \lambda_{\tilde{K}_{\alpha+1}}$ for $\alpha < \tau$, $\tilde{K}_\beta = \bigcap_{\alpha < \beta} \tilde{K}_\alpha$ for all limit ordinals $\beta < \tau$ and $\bigcap_{\alpha < \tau} \tilde{K}_\alpha = \{N\}$.

By the second part of Lemma 32 (applied for $\tilde{K}_{\alpha+1}$ and $\nu = \mu \star \lambda_{\tilde{K}_\alpha}$), this implies $\mu \star \lambda_{\tilde{K}_\alpha} \perp \mu \star \lambda_{\tilde{K}_\beta}$ for $\alpha < \beta < \tau$ (we have $\mu \star \lambda_{\tilde{K}_\beta} \star \lambda_{\tilde{K}_{\alpha+1}} = \mu \star \lambda_{\tilde{K}_{\alpha+1}}$). Then $\tilde{K}_\alpha = K'_\alpha/N$ for compact normal subgroups K'_α of G .

Finally, put $K_\alpha = K'_\alpha K$. By normality, these are compact subgroups. Observe that $\mu \in \mathbf{M}(G/K)$ implies $\mu = \mu \star \lambda_K$, hence using the identifications of Lemma 1 we have $\mu \star \lambda_{\tilde{K}_\alpha} = \mu \star \lambda_{K'_\alpha} = \mu \star \lambda_K \star \lambda_{K'_\alpha} = \mu \star \lambda_{K_\alpha}$. Thus $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_{\alpha+1}} (\neq 0)$ which implies that $K_{\alpha+1}$ is not open in K_α (this need not be true, if one does this construction without μ , resp. $\mu = 0$, and then one would have to remove repetitions, passing to some subfamily). Furthermore, since $K'_\alpha/K'_{\alpha+1} \cong \tilde{K}_\alpha/\tilde{K}_{\alpha+1}$ and $K_\alpha/K_{\alpha+1}$ is homeomorphic to $K'_\alpha/(K'_{\alpha+1}(K \cap K'_\alpha))$, it follows that $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$ for $\alpha < \tau$. It is easy to see that $K_\beta = \bigcap_{\alpha < \beta} K_\alpha$ holds for limit ordinals $\beta < \tau$ and $\bigcap_{\alpha < \tau} K_\alpha = K$. Lemma 18 implies that $\chi(G/K_\alpha) = \chi(G/K_0) + |\alpha|$ when α is infinite. Since $K_0 = N_L K \subseteq L$, we have $\chi(G/K_0) = \chi(G/L)$ and it follows that the K_α will satisfy our demands. \square

Corollary 37. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable.*

- (1) *We have $\bigcap_{L \in \mathcal{K}_K} L = K$ and for every $L \in \mathcal{K}_K$ and every infinite cardinal τ_1 with $\chi(G/L) \leq \tau_1 < \chi(G/K)$ there exists $L' \in \mathcal{K}_K$ with $\chi(G/L') = \tau_1$, $L' \subseteq L$.*
- (2) *For $f \in C_0(G/K)$ we have $\lim_{L \in \mathcal{K}_K} \lambda_L \odot f = f$ in the norm topology. It follows that for $\mu \in \mathbf{M}(G/K)$, $\lim_{L \in \mathcal{K}_K} \mu \star \lambda_L = \mu$ holds in the weak* topology of $\mathbf{M}(G)$, i.e., $\lim_{L \in \mathcal{K}_K} \langle \mu \star \lambda_L, f \rangle = \langle \mu, f \rangle$ for $f \in C_0(G)$.*

Since the embedding of $C_0(G/K)$ (mentioned after Lemma 1) is isometric one can use in (2) either $\|\cdot\|_\infty$ of $C_0(G/K)$ or that of $C_0(G)$. Corresponding limit relations as in (2) hold for any downwards directed family \mathcal{D} of compact subgroups of G satisfying $\bigcap \mathcal{D} = K$.

Proof. Part (1) follows immediately from Lemma 36. If V is a neighbourhood of K in G , it follows by compactness that $L \subseteq V$ holds for some $L \in \mathcal{K}_K$ and (by uniform continuity) this implies convergence of $(\lambda_L \odot f)_{L \in \mathcal{K}_K}$ for $f \in C_0(G/K)$. Then the last statement follows by duality (recall that $\langle \mu, f \rangle = \langle \mu, \lambda_K \odot f \rangle$ and $\lambda_K \odot f \in C_0(G/K)$ for $\mu \in \mathbf{M}(G/K)$, $f \in C_0(G)$). \square

Lemma 38. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable, and $\mu \in \mathbf{M}_{\text{ss}}(G, K)$. There exist two cofinal subsets \mathcal{C} and \mathcal{C}' of \mathcal{K}_K such that $\mu \star \lambda_L \perp \mu \star \lambda_{L'}$ for all $L \in \mathcal{C}$, $L' \in \mathcal{C}'$.*

Proof. Put $\tau = \chi(G/K)$. For the reasons explained in the remark after Lemma 35, we separate two cases.

Case I: τ is a successor cardinal. Let $\mathcal{D} = \{K_\alpha : \alpha < \tau\}$ be a subfamily of \mathcal{K}_K as in Lemma 36. By Lemma 35, \mathcal{D} is cofinal in \mathcal{K}_K . Now we split τ into two cofinal subsets. For example, take

$$\begin{aligned}\mathcal{C} &= \{K_\alpha : \alpha < \chi(G/K), \alpha \text{ is an even ordinal}\} \\ \mathcal{C}' &= \{K_\alpha : \alpha < \chi(G/K), \alpha \text{ is an odd ordinal}\}\end{aligned}$$

(Recall that an ordinal α is called even if $\alpha = \alpha_0 + n$, where n is a finite even number and either α_0 is a limit ordinal or $\alpha_0 = 0$).

Case II: $\chi(G/K)$ is a limit cardinal. We use the following observation: If \mathcal{D} is a family obtained by Lemma 36 and τ_1 is an infinite successor cardinal with $\chi(G/L) < \tau_1 < \tau$, then $\chi(G/K_{\tau_1}) = \tau_1$ and (by Lemma 35) $\mathcal{D}_{\tau_1} = \{K_\alpha : \alpha < \tau_1\}$ is cofinal in $\mathcal{K}_{K_{\tau_1}}$. By Corollary 34, it follows that $\mu \star \lambda_{K_{\tau_1}} \in \mathbf{M}_{\text{ss}}(G, K_{\tau_1})$. Thus (refining Corollary 37(1)) the groups $L \in \mathcal{K}_K$ for which $\mu \star \lambda_L \in \mathbf{M}_{\text{ss}}(G, L)$ form a cofinal subset of \mathcal{K}_K and all infinite successor cardinals ($< \tau$) thereby arise as $\chi(G/L)$.

Now, we split the set of cardinals $< \tau$ into two cofinal subsets. Combined, we might take for example,

$$\begin{aligned}\mathcal{C} &= \{L \in \mathcal{K}_K : \chi(G/L) \text{ is an even cardinal and } \mu \star \lambda_L \in \mathbf{M}_{\text{ss}}(G, L)\} \\ \mathcal{C}' &= \{L \in \mathcal{K}_K : \chi(G/L) \text{ is an odd cardinal and } \mu \star \lambda_L \in \mathbf{M}_{\text{ss}}(G, L)\}\end{aligned}$$

(Recall that a cardinal $\tau_1 = \aleph_\alpha$ is called even if α is an even ordinal). Then cofinality of $\mathcal{C}, \mathcal{C}'$ easily follows. By definition, $\mathbf{M}_{\text{ss}, \tau_1}(G) \perp \mathbf{M}_{\text{ss}, \tau_2}(G)$ holds for $\tau_1 \neq \tau_2$. This gives $\mu \star \lambda_L \perp \mu \star \lambda_{L'}$ for all $L \in \mathcal{C}, L' \in \mathcal{C}'$. \square

Proposition 39. *Let G be a locally compact group, K a compact subgroup of G such that G/K is non-metrizable. If $\mu \in \mathbf{M}(G/K)$ but $\mu \notin \mathbf{M}_{\text{ai}}(G, K)$, then the net $(\mu \star \lambda_L)_{L \in \mathcal{K}_K}$ is not convergent for the weak* topology of $\mathbf{M}(G)^{**}$, i.e., there exists $h \in \mathbf{M}(G)^*$ such that $(\langle h, \mu \star \lambda_L \rangle)_{L \in \mathcal{K}_K}$ does not converge.*

Proof. Decomposing μ by Theorem 22 (iii) and applying Lemma 33 to the \mathbf{M}_{ai} -component, we may assume that $\mu \in \mathbf{M}_{\text{ss}}(G, K)$. Since $\mu \neq 0$ there exists $f \in C_0(G/K)$ such that $\langle \mu, f \rangle \neq 0$. By Corollary 37, we have $\lim_{L \in \mathcal{K}_K} \langle \mu \star \lambda_L, f \rangle = \langle \mu, f \rangle$ and the same limit arises for any cofinal subfamily of \mathcal{K}_K . Now choose subsets \mathcal{C} and \mathcal{C}' as in Lemma 38. By Lemma 6 there exists $h \in \mathbf{M}(G)^*$ such that $h = f$ on the linear

subspace generated by $\{\mu \star \lambda_L : L \in \mathcal{C}\}$ and $h = -f$ on the linear subspace generated by $\{\mu \star \lambda_L : L \in \mathcal{C}'\}$ (with $\|h\| \leq \|f\|$). \square

Theorem 40. *Let G be a locally compact group and K a compact subgroup of G . Then $Z_t(\mathbf{M}(G)^{**}) \cap \mathbf{M}(G/K)^{**} \subseteq \mathbf{M}(G/K)$.*

The case of the trivial subgroup $K = \{e_G\}$ gives the Main Theorem.

Proof. We will use induction on $\tau = \chi(G/K)$. The case where G/K is metrizable (i.e. $\chi(G/K) \leq \aleph_0$) has been settled in Theorem 17. Thus we can assume that G/K is non-metrizable and that the theorem holds for all subgroups $L \in \mathcal{K}_\tau^\circ$. Put $M_0 = \mathbf{M}_{\text{ss}}(G, K)$, $M_1 = \mathbf{M}_{\text{ai}}(G, K)$, $M_2 = \mathbf{M}(G/K)$ and $\widetilde{M}_0 = \mathbf{M}_{\text{ss}, \tau}$, $\widetilde{M}_1 = \mathbf{M}_{\text{ai}, \tau}$, $\widetilde{M}_2 = \mathbf{M}_\tau$. Then Theorem 12 and Corollary 31 show that

$$Z_t(\mathbf{M}(G)^{**}) \cap \mathbf{M}(G/K)^{**} \subseteq \mathbf{M}_{\text{ss}}(G, K) \oplus \mathbf{M}_{\text{ai}}(G, K)^{**}.$$

Thus (since $\mathbf{M}(G) \subseteq Z_t(\mathbf{M}(G)^{**})$) it will be enough to consider the case where $\mathfrak{m} \in Z_t(\mathbf{M}(G)^{**}) \cap \mathbf{M}_{\text{ai}}(G, K)^{**}$.

By Lemma 1, $\mathbf{M}(G/K) = \mathbf{M}(G) \star \lambda_K$. Under the standard embedding of the bidual (see the beginning of Section 3), it follows easily (using weak* density and continuity) that $\mathbf{M}(G/K)^{**} = \mathbf{M}(G)^{**} \square \lambda_K$ holds. $Z_t(\mathbf{M}(G)^{**})$ being a subalgebra, the inductive assumption implies that $\mathfrak{m} \square \lambda_L = \mu_L \in \mathbf{M}(G/L)$ for all $L \in \mathcal{K}_\tau^\circ$. Let $\mu \in \mathbf{M}(G/K)$ be the measure obtained by restricting \mathfrak{m} to $C_0(G/K)$ (subspace of $\mathbf{M}(G/K)^*$). We get $\langle \mu_L, f \rangle = \langle \mathfrak{m} \square \lambda_L, f \rangle = \langle \mu, \lambda_L \odot f \rangle = \langle \mu \star \lambda_L, f \rangle$ for all $f \in C_0(G/K)$. Hence $\mu_L = \mu \star \lambda_L$. Now let $\bar{\delta} \in \mathbf{M}_{\text{ai}}(G, K)^{**}$ be a weak* accumulation point of the net $(\lambda_L)_{L \in \mathcal{K}_K}$. By Lemma 33, $(\lambda_L)_{L \in \mathcal{K}_K}$ is a right approximate unit for $\mathbf{M}_{\text{ai}}(G, K)$. Since $\mathfrak{m} \in Z_t(\mathbf{M}(G)^{**})$ implies $\mathfrak{m} \in Z_t(\mathbf{M}_{\text{ai}}(G, K)^{**})$, it follows that $\mathfrak{m} \square \bar{\delta} = \mathfrak{m}$ (see [Da] Prop. 2.9.16 and its proof). Then from $\mu \in \mathbf{M}(G) \subseteq Z_t(\mathbf{M}(G)^{**})$, we get, by using an appropriate refinement of the net $(\lambda_L)_{L \in \mathcal{K}_K}$: $\mathfrak{m} \square \bar{\delta} = \lim \mathfrak{m} \square \lambda_L = \lim \mu_L = \lim \mu \star \lambda_L = \mu \square \bar{\delta}$. Hence $\mathfrak{m} = \mu \square \bar{\delta}$. Since this holds for every accumulation point $\bar{\delta}$, it follows that $\mathfrak{m} = \lim_{L \in \mathcal{K}_K} \mu \star \lambda_L$ (weak* limit in $\mathbf{M}(G)^{**}$). Then Proposition 39 implies $\mu \in \mathbf{M}_{\text{ai}}(G/K)$ and by Lemma 33 we get $\mathfrak{m} = \mu \in \mathbf{M}(G/K)$. \square

Final Remarks.

(a) There are examples where $\mathbf{M}(G/K)$ is *not* strongly Arens irregular (notation as in Theorem 40). It turns out that for non-commutative G the left topological centre $Z_t^{(1)}$ and the right topological centre $Z_t^{(2)}$ ([DL] Def. 2.17) need not coincide. One

can show in a similar way as in the proof of Theorem 40 that $Z_t^{(2)}(\mathbf{M}(G/K)^{**}) = \mathbf{M}(G/K)$ holds for all compact subgroups K of a locally compact group G (this can be seen as an example for the more general approach of [left] group actions, as sketched in Remark (b) after Corollary 31).

Furthermore, there is the space $G//K$ of double cosets (see [Jw], in [Di, 22.6]) it is denoted as $K \backslash G / K$. As in Lemma 1, we may identify $\mathbf{M}(G//K)$ with the subalgebra $\lambda_K \star \mathbf{M}(G) \star \lambda_K$ of $\mathbf{M}(G, K) = \mathbf{M}(G) \star \lambda_K \subseteq \mathbf{M}(G)$ and it is not hard to see that $Z_t^{(1)}(\mathbf{M}(G//K)^{**}) = Z_t^{(1)}(\mathbf{M}(G/K)^{**}) \cap \mathbf{M}(G//K)^{**}$. Consider now $G = SL(2, \mathbb{C})$, $K = SU(2)$ (a maximal compact subgroup). Then one can show that $Z_t^{(1)}(\mathbf{M}(G//K)^{**}) \supsetneq \mathbf{M}(G//K)$ and this implies that

$$Z_t^{(1)}(\mathbf{M}(G/K)^{**}) \supsetneq \mathbf{M}(G/K) \quad (= Z_t^{(2)}(\mathbf{M}(G/K)^{**})).$$

Thus $\mathbf{M}(G//K)$ and $\mathbf{M}(G/K)$ are not strongly Arens irregular (in this example). On the other hand, one can show that $Z_t^{(1)}(\mathbf{M}_{\mathbf{a}}(G//K)^{**}) = \mathbf{M}_{\mathbf{a}}(G//K)$ and $Z_t^{(1)}(\mathbf{M}_{\mathbf{a}}(G/K)^{**}) = \mathbf{M}_{\mathbf{a}}(G/K)$. Thus $\mathbf{M}(G//K)$ and $\mathbf{M}(G/K)$ are not Arens regular (in this example). The convolution structure of $\mathbf{M}(G//K)$ is described explicitly in [Jw, 15.5]. In particular, one has $\mu \star \nu \in \mathbf{M}_{\mathbf{a}}(G//K)$ for all $\mu, \nu \in \mathbf{M}(G//K)$ with $\mu(\{K\}) = \nu(\{K\}) = 0$ (showing a big difference to the abelian case). It follows that for $\mu \in \mathbf{M}(G//K)$ (with $\mu \neq 0$, $\mu(\{K\}) = 0$) there is no analogue of Theorem 15 when replacing translates $\mu \star x$ by “generalized translates” $\mu \star \nu_x$ with $\nu_x = \lambda_K \star \delta_x \star \lambda_K$. More details will be given elsewhere.

(b) In most of the paper the focus has been on locally compact groups and many proofs made heavy use of local compactness. We want to sketch here another approach which allows to treat some classes of non-locally compact groups.

First observe that if N is a nowhere dense subset of a topological group G , then (since group multiplication is an open mapping) $\{(x, y) : xy^{-1} \in N\}$ must be nowhere dense in $G \times G$.

Assume now that G is a Polish group. Then one can apply a result of Mycielski ([M, Theorem 1]) and it follows that given a non-empty meagre subset Z of G , there exists a perfect set P in G such that $xy^{-1} \notin Z$ for all distinct x and y in P . Next observe that if G is *not* locally compact then every compact subset must be nowhere dense. Hence, if A is any σ -compact subset of G , then $Z = A^{-1}A$ is meagre. Applying the result above, there exists a perfect set P in G such that the sets Ax ($x \in P$) are pairwise disjoint. Now, if μ is an arbitrary finite Borel measure on a non-locally compact Polish group G , then (being a Radon measure) it

is concentrated on a σ -compact subset and it follows that μ is 2^{\aleph_0} -thin (recall that $|P| = 2^{\aleph_0}$ for perfect subsets).

For every uncountable Polish group, $|\mathbf{M}(G)| = 2^{\aleph_0}$ (the algebra of sets generated by a countable basis of the topology is again countable and by elementary measure theory, a finite, σ -additive set function on a σ -algebra is uniquely determined by its values on a generating subalgebra), in particular $d(\mathbf{M}(G)) = 2^{\aleph_0}$. We can apply now the arguments of Section 3 and 4 (actually, local compactness is not needed there) and it follows that $Z_t(\mathbf{M}(G)^{**}) = \mathbf{M}(G)$ holds for every Polish group G . Thus $\mathbf{M}(G)$ is strongly Arens irregular for Polish groups.

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